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Models for the Brane-Bulk Interaction: Toward Understanding Braneworld Cosmological Perturbations

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Abstract

Using some simple toy models, we explore the nature of the brane-bulk interaction for cosmological models with a large extra dimension. We are in particular interested in understanding the role of the bulk gravitons, which from the point of view of an observer on the brane will appear to generate *dissipation* and *nonlocality*, effects which cannot be incorporated into an effective (3+1)-dimensional Lagrangian field theoretic description. We explicitly work out the dynamics of several discrete systems consisting of a finite number of degrees of freedom on the boundary coupled to a (1+1)-dimensional field theory subject to a variety of wave equations. Systems both with and without time translation invariance are considered and moving boundaries are discussed as well. The models considered contain all the qualitative feature of quantized linearized cosmological perturbations for a Randall-Sundrum universe having an arbitrary expansion history, with the sole exception of gravitational gauge invariance, which will be treated in a later paper.

I. INTRODUCTION

In this paper we consider the problem of computing the cosmological perturbations for the one-brane Randall Sundrum model [1, 2] in which we inhabit a (3+1)-dimensional brane embedded in a pure (4+1)-dimensional anti de Sitter bulk spacetime. A Z_2 symmetry about the brane is supposed and the perturbations are treated to linear order. The only bulk degrees of freedom are the (4+1)-dimensional bulk gravitons or gravity waves, which propagate in the bulk [1, 3]. These are emitted, absorbed, and reflected by the brane as well as reflected, or perhaps, more properly, diffracted by the bulk, making the problem of predicting the cosmological perturbations in a braneworld scenario much more complex than its $(3+1)$-dimensional counterpart.

A vast literature exists on the problem of calculating cosmological perturbations in a braneworld scenario. Some progress has been made, either by resorting to various approximations or by considering special spacetimes on the brane, such as eternal pure de Sitter space, for which additional symmetry can be exploited. A non-exhaustive sampling of the literature can be found in ref. [4]–[8] and the references cited therein. In this paper we develop some techniques that could be employed to solve the problem completely, for a general expansion history and without resort to any approximations. The work reported here, which is most closely related to considerations in Gorbunov, Rubakov and Sibiryakov [5] and the approach presented in the series of papers by Mukoyama [6], constitutes a first step toward this goal. Although we only discuss the connection with the Randall-Sundrum cosmologies, the methods and ideas developed here would apply equally well to more complex and perhaps more realistic situations with additional degrees of freedom in the bulk.

It is this coupling between the brane and bulk degrees of freedom that renders the problem of cosmological perturbations in the braneworld scenario difficult. For linearized cosmological perturbations in the ordinary (3+1)-dimensional cosmology, the problem may be *diagonalized* by expanding the three spatial dimensions in Fourier components $\exp[i\mathbf{k}\cdot\mathbf{x}]$ (where for simplicity here we assume a spatially flat universe) [9]. To linear order, there is no mixing between different $\mathbf{k}$. Consequently, each such block may be analysed separately. The equations may be diagonalized further by separating the *scalar, vector,* and *tensor*

sectors. Here we define any quantity expressible in terms of derivatives acting on a scalar as *scalar*. Likewise, any quantity expressible as derivatives acting on a pure *vector* potential is regarded *vector,* etc. Under these definitions, spatial differentiation does not mix these three sectors, giving the evolution equations a block diagonal form. For each $\mathbf{k}$ there is a finite number of degrees of freedom whose time evolution is described by a finite number of ordinary coupled differential equations in cosmic time t.

In the presence of an extra dimensional "bulk," the number of degrees of freedom in each $\mathbf{k}$ sector is greatly—in fact, infinitely—enlarged. (See, for example, the review [7]). When the size of the extra dimension is finite, the additional bulk degrees of freedom are discretely infinite, their spacing inversely proportional to the size of the extra dimension. When the extra dimension is very small, a large mass gap must be overcome in order to access the infinite tower of bulk excitations and at low energies the (3+1)-dimensional limit is obtained. However, when the extra dimension is infinite, the bulk degrees of freedom are continuous, labeled by an index k_5, where $0 \leq k_5 < +\infty$. For the case of a warped bulk spatial geometry, it has been shown how the coupling to the brane of the lowest energy degrees of freedom is suppressed, giving on large scales ordinary (3+1)-dimensional gravity on the brane. However, we would like to explore the corrections to such a limit. We would like to discover any possible differences in the predictions for the cosmological perturbations between the standard (3+1)-dimensional cosmology and a braneworld cosmology. Consequently, we must solve the interacting brane-bulk system.

The vast enlargement of the number of degrees of freedom renders the problem difficult in two respects: Firstly, the equations become more complicated. But this is not all. Secondly, a solution to the problem of braneworld cosmological perturbations requires specifying initial conditions for the infinite number of extra degrees of freedom. While the first problem is of a technical, computational character, the second is of a more fundamental or physical nature.

The problem of initial conditions for the bulk degrees of freedom can be further elucidated by considering the Penrose diagram for Randall-Sundrum Universes with various expansion histories, as indicated in Fig. 1. Panel (a) indicates the patch on the conformal diagram covered by the standard Randall-Sundrum coordinates, having the line element

$$ds^2 = dx_5^2 + \exp[-2x_5/\ell] \cdot \Big[-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\Big]. \tag{1}$$

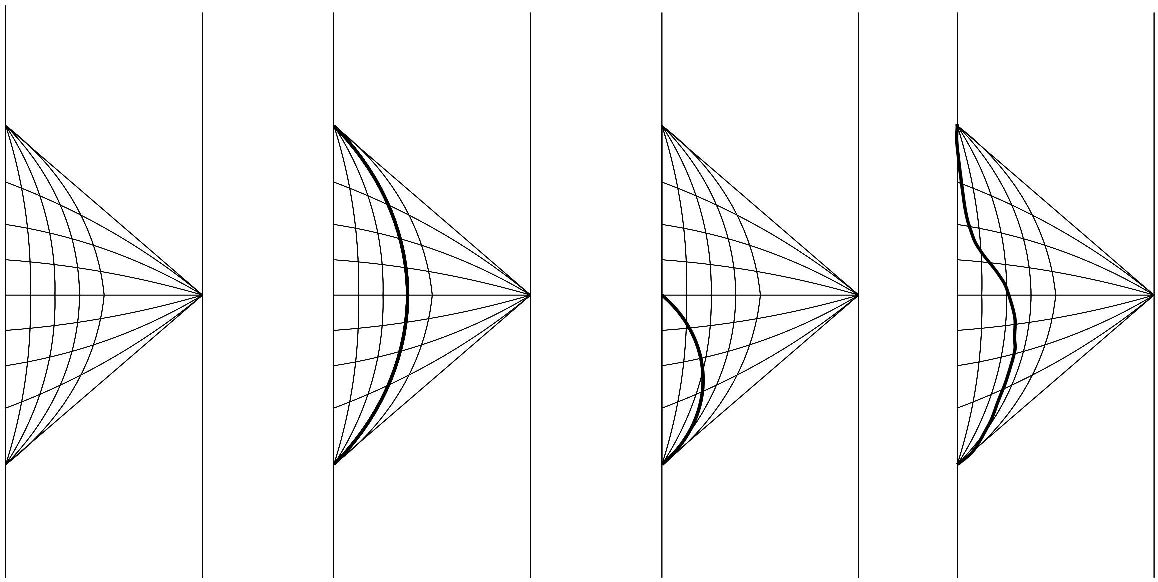

FIG. 1: **Various braneworld cosmologies embedded into the conformal diagram for maximally-extended AdS^5.** Panel (a) shows the triangular portion of maximally-extended AdS covered by the Randall-Sundrum coordinates. Time runs upward and the horizontal direction represents the fifth dimension, the other three transverse dimensions being orthogonal to the page. The family of lines focusing to the vertex on the right represents surfaces of constant Randall-Sundrum time. These asymptote in the future and in the past to the null surfaces $H_{(+)}$ and $H_{(-)}$, which are the future and past Cauchy horizons of this triangular patch. The vertical curves connecting the lower vertex to the upper vertex of the triangular regions represent surfaces where the Randall-Sundrum coordinate for the fifth dimension is constant. These are also surfaces on which the scale factor $a(x_5)$ for the three transverse spatial dimensions is constant. As one passes from right to left this scale factor increases. In panel (b) the trajectory of the brane (indicated by the heavy timelike curve) for a static Randall-Sundrum universe, having a Minkowski induced geometry, is indicated. In panels (c) and (d) we consider expanding universes, where the motion of the brane to the left (with respect to the surfaces of constant x_5) implies expansion. In panel (c) the brane worldline for a de Sitter induced geometry on the brane is illustrated. The brane emanates from the lower vertex, as for the static brane, but because of the rapidity of the expansion, strikes the boundary at conformal infinity before reaching the upper vertex. In other words, de Sitter proper time on the brane becomes infinite at finite Randall-Sundrum time. Panel (d) illustrates a braneworld universe that is initially inflating but whose expansion later slows down to become a dust-dominated universe. In this case, because of the deceleration, future infinity on the brane corresponds to infinite Randall-Sundrum time. In all cases there is a past Cauchy horizon $H_{(-)}$ on which initial data for the bulk modes must be specified to completely determine the subsequent evolution of the coupled brane-bulk system.

Panel (b) indicates a static M^5 brane universe. Here the brane trajectory, indicated by the heavy curve, coincides with one of the x_5 constant surfaces. $H_{(+)}$ and $H_{(-)}$ are the bulk horizons with respect to the observers on the brane. Panel (c) indicates a Randall-Sundrum universe with a dS^5 geometry on the brane, and panel (d) indicates a universe that is initially inflationary but then reheats to become a decelerating (e.g., dust-dominated) universe. In all cases, there is the past horizon $H_{(-)}$, of particular interest to us, where

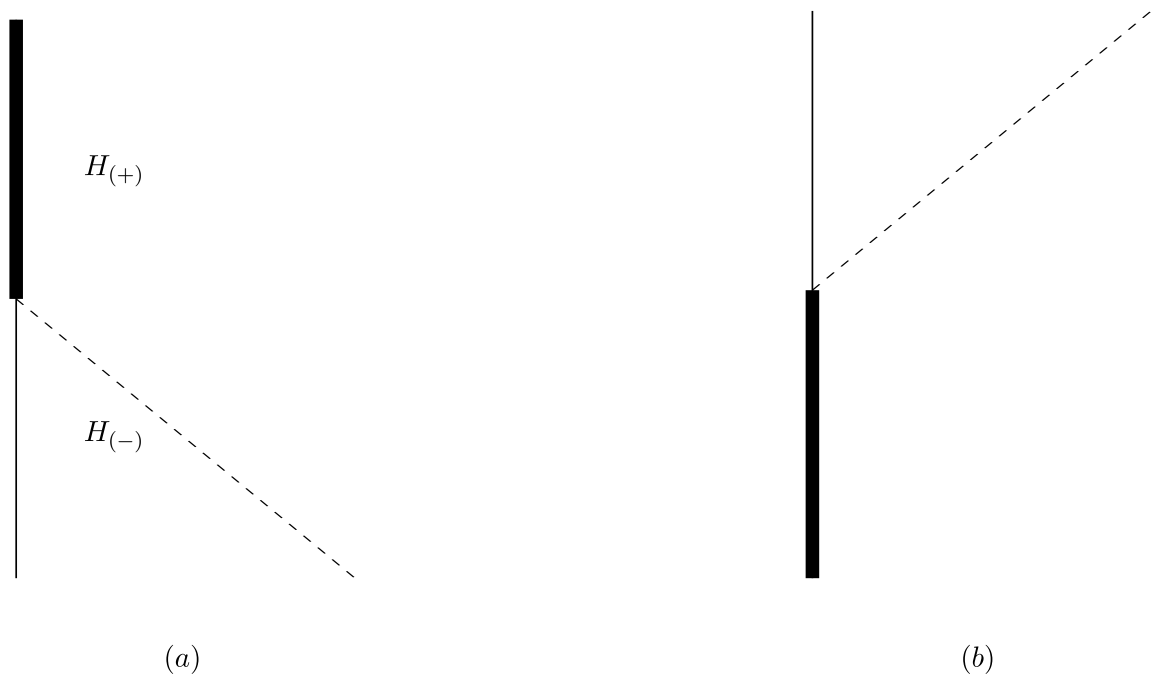

FIG. 2: **Fundamental processes for the bulk-brane interaction.** The thin vertical solid line indicates the position of the brane and the bulk is situated to its right. A bulk graviton, indicated by the dashed line, may strike the brane, annihilating itself and creating an excitation of the brane, indicated by the heavy vertical line, as illustrated in panel (a). Likewise, an excitation of the brane may decay, emitting a bulk graviton, as illustrated in panel (b). Because we are interested here only in the linearized theory, there are no vertices joining more than two external lines. Propagation in the bulk may involve scattering and is not in general along straight trajectories. Moreover, bulk gravitons may reflect off the boundary without changing the excitations on the brane.

initial data for the bulk gravitons must be specified.

An initial state for the brane-bulk system may be characterized completely by specifying the quantum state of the incoming gravitons on $H_{(-)}$ and of the degrees of freedom on the brane at the intersection of the brane with $H_{(-)}$. Subsequently, the bulk and the brane degrees of freedom interact. The relevant fundamental processes are illustrated in Fig. 2, As shown in panel (a), bulk gravitons may be absorbed and transformed into quanta on the brane. Similarly, as shown in panel (b), brane excitations may decay through the emission of bulk gravitons. These may either escape to future infinity, leading to a sort of dissipation, or be re-absorbed by the brane, leading to what from the four-dimensional point of view appears as *nonlocality.* Such *dissipation* and *nonlocality* are effects that cannot be incorporated into an effective four-dimensional effective field theory description, and it is hoped that by studying these phenomena one might be able to find some distinctive observational signature for the presence of extra dimensions.

In this article we present some simple toy models where a single or finite number of

degrees of freedom is coupled to a one-dimensional continuum of degrees of freedom. Such (1+1)-dimensional field theories coupled to a boundary having its own dynamics are the analogue of a fixed **k**-sector of the brane-bulk system. The degrees of freedom on the boundary represent the degrees of freedom on the brane and the degrees of freedom of the one-dimensional continuum represent the degrees of freedom of the bulk. We consider a number of examples of increasing complexity that exhibit all the qualitative features of the brane-bulk system except those concerning gravitational gauge invariance and gauge fixing. These questions shall be treated in a future publication.

In Section II we study the simplest such system. A harmonic oscillator is coupled to a stretched string so that excitations of the oscillator may decay, or dissipate, emitting waves on the string that propagate to infinity. If there are no incoming waves on the string, which is classically possible but not quantum mechanically, the motion of the oscillator is described exactly by the equation of motion for the dampened harmonic oscillator. Similarly, incoming waves may excite the oscillator. In the analogy to braneworld cosmology, the oscillator represents a degree of freedom on the brane and the string represents the bulk gravitons. We work out the quantum mechanical description of this system and in particular of its effective description in detail. We show how the annihilation and creation operators localized at the end of the string, $a_{osc}(t)$ and $a^\dagger_{osc}(t)$, do not have definite frequency but rather are the superposition of modes of different frequency having a certain characteristic width of order the classical decay rate γ. It is this spread in frequency that causes the commutator $[a_{osc}(t), a^\dagger_{osc}(t')]$ to decay in modulus when $|t-t'|$ becomes large compared to γ^{-1}. We also consider several oscillators coupled to each other and to a string and the case where the mass density of the string is not uniform so that there are reflections, leading to nonlocality in the effective description for the oscillator degree of freedom. All the systems in this section have a time-translation invariance, so that a Fourier decomposition in time can be employed, greatly simplifying the problem.

In Section III we consider cases where there is no time translation invariance, in which the coupling evolves with time. While the examples of the previous section are analogous to the static Randall-Sundrum brane, where the brane geometry is that of (3+1)-dimensional Minkowski space, the examples of section III are more akin to the expanding braneworld universe, where the couplings of the various modes to the brane evolve with time. In the

time translation invariant case, at any given finite time, the quantum state of a mode depends only on the quantum state in the bulk at past infinity. However, with the time-dependent interactions, cases may be contemplated where the state on the brane depends on a linear combination of the initial state on the brane and that in the bulk. Some explicit examples are worked out where the linear canonical transformation—a sort of S matrix between the "in" and "out" states—is calculated. The matrix linking the "in" and "out" states for the bulk is akin to the calculation in Gorbunov, Rubakov and Sibiryakov [5] where the Bogoliubov transformation between the "in" and "out" states is calculated for a brane having a dS^4 geometry due to a pure cosmological constant and nothing else on the brane.

In section IV we consider the complications that arise when the wave equation on the string is such that the general solution can no longer be decomposed into left and right movers. Any sort of mass term or non-uniformity in the speed of propagation will render such a decomposition impossible by causing disturbances emitted from the brane to scatter and propagate back onto the boundary. There can be many such multiple reflections. In the case of interest to us, the fact that the bulk geometry is AdS^5 rather than M^5 causes such scatterings, which might also be described as *diffraction,* particularly for wavelengths of order the AdS curvature scale. We develop a perturbative approach to treating such scattering. The final result indicates how the bulk interaction in the toy model can be expressed in terms of integral kernels. The approach is similar to that sketched in the papers of Mukoyama [6].

Finally, in section V we consider the case where the boundary follows an arbitrary given timelike trajectory $x(t)$. (Here x_5 is simply shortened to the x of our toy models.) We show how the relevant Green's functions may be constructed using virtual sources whose strengths are determined by solving a Volterra integral equation of the second type. These techniques are relevant to the brane-bulk problem because the propagation of the bulk gravitons is most trivial in a coordinate system where the bulk is static but the brane moves as a result of the expansion of the universe. In section VI we conclude with some closing comments.

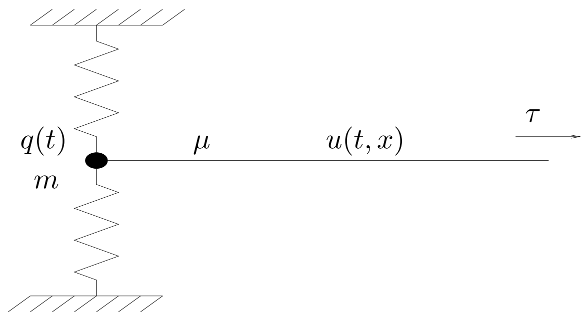

FIG. 3: **Simple model for the brane-bulk interaction.** A simple harmonic oscillator consisting of a mass m bound to a spring of stiffness $k = m\bar{\omega}^2$ is coupled to a stretched string of linear density μ and tension τ. $q(t)$ represents the vertical displacement of the oscillator, and the field $u(t,x)$ represents the vertical displacement of the stretched string. Here the oscillator represents a degree of freedom localized on the brane and the modes on the string represent the bulk gravitons. As the mass oscillates, energy escapes into the bulk, leading to dissipation. If the linear density of the string is not uniform, some of these modes scatter back, leading to effects nonlocal in time.

II. EXAMPLES WITH TIME TRANSLATION INVARIANCE

A. Quantum Mechanics With Dissipation : A Single Mass Coupled to a String

After the linearized perturbations of the brane-bulk system have been decomposed into their Fourier components with respect to three transverse spatial dimensions, the evolution equations decompose into independent blocks, each labeled by the three-dimensional wave number **k**. Each such block has the dynamics of the $(1+1)$-dimensional field theory coupled to one or more degrees of freedom localized on the brane boundary. In this section, we study a simple mechanical model of such a block. We consider a simple harmonic oscillator attached to the end of a string (see Figure 3). In the absence of coupling to the string, the equation of motion for the oscillator is

$$m\ddot{q}(t) + m\bar{\omega}^2 q(t) = F(t) \tag{2}$$

where the external driving force $F(t)$ vanishes. The quantum states are simply the usual simple harmonic oscillator ladder of states of energy $E_n = \hbar\bar{\omega}(n + \frac{1}{2})$.

We now introduce dissipation in a 'conservative' way—that is, we couple the above oscillator to a system with an infinite number of degrees of freedom in such a way that a classically excited state of the oscillator progressively radiates its energy into the continuum and this energy never returns. (Some other approaches to treating dissipation within the framework of quantum mechanics may be found in refs. [10].) For concreteness, consider a

string of linear density μ and tension τ. For simplicity, we set $\mu = \tau$ for the string, so that

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0. \tag{3}$$

If we attach the string to the oscillator at $x = 0$, the driving force exerted by the string on the mass becomes

$$F(t) = \tau \frac{\partial u}{\partial x}\bigg|_{x=0}. \tag{4}$$

Classically, we solve this system by postulating as an initial condition the absence of any incoming (or left-moving) waves on the string. Using the ansatz

$$u(t,x) = C \; \exp[i\omega(x-t)] \tag{5}$$

and setting $\gamma = \tau/m$, we obtain from (2) where $q(t) = u(t, x = 0)$ the quadratic equation

$$\bar{\omega}^2 - \omega^2 = i\gamma\omega, \tag{6}$$

having the solution

$$\omega = \frac{-i\gamma}{2} \pm \sqrt{\bar{\omega}^2 - \frac{\gamma^2}{4}}. \tag{7}$$

This solution, of course, diverges at large positive x and large negative t; however, these divergences are unphysical, because in constructing a Green's function, we match to a vanishing solution below a line of $(t - x) < t'$ where t' is the time corresponding to the singular source located at $x = 0$ at the oscillator. Classically, from an effective point of view, we could dispense with the string and its degrees of freedom infinite in number and instead simply use the equation

$$m\Big[\ddot{q}(t) + \gamma\dot{q}(t) + \bar{\omega}^2 q(t)\Big] = F_{ext}(t) \tag{8}$$

where the force exerted by the string on the oscillator at $x = 0$, $F_{string}(t)$, has been placed on the left-hand side and is now the dissipative term $m\gamma\dot{q}(t)$, and $F_{ext}(t)$ is included to denote an external force on the oscillator other than that exerted by the string. The arrow of time here was set by our initial condition that there be no incoming waves on the string in the past. If we had instead postulated no incoming waves in the future, the sign of the $\gamma\dot{q}$ term would be inverted. It is important to note that, from a classical point of view, eqn. (8) is exact. There are no corrections.

Quantum mechanically, the initial condition above does not make sense. Vacuum fluctuations of the string require the presence of at least some left-moving waves. Otherwise, it would not be possible to satisfy the canonical commutation relations. Consequently, we must proceed by some other means. We start by postulating a vacuum state for the string-oscillator system. Note that when coupled to the string in the above way, the oscillator degree of freedom disappears. It has become part of the string, because we have set $q(t) = u(t, x = 0)$.[1]

For a fixed time dependence proportional to $e^{-i\omega t}$, we find the normal modes of the string with the ansatz

$$u_\omega(t, x) = \sin\Big[\omega x - \phi(\omega)\Big]\ e^{-i\omega t}, \tag{9}$$

where ω is given in (7). We have

$$\Big(\bar{\omega}^2 - \omega^2\Big) u_\omega(t, x = 0) = \gamma \frac{\partial u_\omega}{\partial x}(t, x = 0), \tag{10}$$

or equivalently

$$\Big(\bar{\omega}^2 - \omega^2\Big) \sin\Big[\phi(\omega)\Big] = -\gamma\omega \cos\Big[\phi(\omega)\Big], \tag{11}$$

where $\gamma = (\tau/m)$. It follows that

$$\sin\Big[\phi(\omega)\Big] = \frac{-\gamma\omega}{\sqrt{\gamma^2\omega^2 + (\omega^2 - \bar{\omega}^2)^2}} \approx \frac{-(\gamma/2)}{\sqrt{(\gamma/2)^2 + (\omega - \bar{\omega})^2}}, \tag{12}$$

where the latter approximation holds when $\gamma \ll \bar{\omega}$, in other words when the quality factor of the oscillator is very large.

The field on the string may be expanded as

$$\hat{u}(t, x) = \int_0^{+\infty} \frac{d\omega}{\sqrt{2\pi\mu\omega}} \sin\Big[\omega x - \phi(\omega)\Big] \Big[\hat{a}(\omega)\ e^{-i\omega t} + \hat{a}(\omega)^\dagger\ e^{+i\omega t}\Big] \tag{13}$$

where

$$[\hat{a}(\omega), \hat{a}(\omega')^\dagger] = \delta(\omega - \omega'). \tag{14}$$

[1] To understand how this occurs, it is useful to consider the string as a continuum limit of a series of small masses $(\Delta m) = \mu(\Delta l)$ connected by springs of separation (Δl) attached to a harmonically bound mass of fixed magnitude M at the end point. In the continuum limit, the frequency of the mode where the large mass oscillates relative to its neighbors approaches infinity. It is always of energy of order the cutoff.

The normalization factor follows from the canonical commutation relations for the field $\hat{u}(t,x)$. It follows that the quantum operator corresponding to the oscillator degree of freedom localized at the end of the string at time t is given by

$$\begin{aligned}\hat{q}(t) &= -\int_0^{+\infty} \frac{d\omega}{\sqrt{2\pi\mu\omega}} \sin\Big[\phi(\omega)\Big]\Big[\hat{a}(\omega)\ e^{-i\omega t} + \hat{a}(\omega)^\dagger\ e^{+i\omega t}\Big] \\ &= \int_0^{+\infty} \frac{d\omega}{\sqrt{2\pi m\gamma\omega}} \frac{\gamma\omega}{\sqrt{\gamma^2\omega^2 + (\omega^2-\bar{\omega}^2)^2}}\Big[\hat{a}(\omega)\ e^{-i\omega t} + \hat{a}(\omega)^\dagger\ e^{+i\omega t}\Big].\end{aligned} \tag{15}$$

Define

$$\rho^{1/2}(\omega) = N\ \sqrt{\frac{\gamma\omega}{\gamma^2\omega^2 + (\omega^2-\bar{\omega}^2)^2}} \tag{16}$$

where

$$N^{-2} = \int_0^{+\infty} d\omega\ \frac{\gamma\omega}{\gamma^2\omega^2 + (\omega^2-\bar{\omega}^2)^2}. \tag{17}$$

We may define

$$\hat{a}_{osc}(t) = \int_0^{+\infty} d\omega\ \rho^{1/2}(\omega)\ \ \exp[-i\omega t]\ \hat{a}(\omega). \tag{18}$$

Because

$$\int_0^{+\infty} d\omega\ \rho(\omega) = 1, \tag{19}$$

it follows that at equal times

$$\Big[\hat{a}_{osc}(t), \hat{a}^\dagger_{osc}(t)\Big] = 1, \tag{20}$$

as for an ordinary harmonic oscillator. However, in the general case

$$\Big[\hat{a}_{osc}(t), \hat{a}^\dagger_{osc}(t')\Big] = \int_0^{+\infty} d\omega\ \rho(\omega)\ \ \exp[-i\omega(t-t')], \tag{21}$$

which for unequal times is of a modulus smaller than one. For $|t-t'| \ll \gamma^{-1}$ this is to a good approximation a simple phase, namely $\exp[-i\bar{\omega}(t-t')]$, just as one would obtain by ignoring the dampening and pretending that the oscillator is uncoupled with respect to the string. However, when $|t-t'|$ becomes comparable to or greater than γ^{-1}, one observes a sizable diminution in modulus. Unless the density $\rho(\omega)$ has some delta functions spikes, the Riemann-Lebesgue lemma implies that, for times sufficiently well separated, the operators almost commute.

Let us consider the special case of a uniform string of small tension, so that the approximation on the far right-hand side of eqn. (12) holds where $\gamma \ll \bar{\omega}$. In this case

$$\rho(\omega) = \frac{1}{\pi}\frac{(\gamma/2)}{(\gamma/2)^2 + (\omega-\bar{\omega})^2} \tag{22}$$

and with very little error we may replace the interval of integration $[0,+\infty)$ with the doubly-infinite interval $(-\infty,+\infty)$, so that

$$\begin{aligned}\left[\hat{a}_{osc}(t),\hat{a}^{\dagger}_{osc}(t)\right] &\approx \frac{1}{\pi}\int_{-\infty}^{+\infty} d\omega\ \frac{(\gamma/2)}{(\gamma/2)^2+(\omega-\bar{\omega})^2}\ \exp[-i\omega(t-t')] \\ &= \exp[-i\bar{\omega}(t-t')]\cdot\exp[-(\gamma/2)|t-t'|]. \end{aligned} \tag{23}$$

For broad resonances, there are corrections to this formula, but the general qualitative behavior persists.

B. Multiple reflections and non-locality

We may also contemplate more complicated systems for which the classical behavior of the oscillator is no longer local in time, as for example as indicated in eqn. (8), but rather an equation of the form

$$\ddot{q}(t)+\gamma\dot{q}(t)+\bar{\omega}^2 q(t) = \frac{1}{m}\left[F_{ext}(t)+\int_{-\infty}^{t} dt'\ G(t-t')\ F_{ext}(t')\right] \tag{24}$$

where the kernel $G(t-t')$ represents the amplitude for a wave on the string generated at time t' reflecting back to strike the oscillator subsequently at time t. Such nonlocal behavior will arise in the brane-bulk system because bulk gravitons initially emitted by the brane will be scattered back to the brane by the curved bulk geometry.

For concreteness, consider the case where rather than being uniform, the linear density of the string abruptly increases at $x=L$, so that at that point the propagation speed suddenly drops from c_l to c_r as one passes from the left to the right. Physically, a portion of the outgoing wave is transmitted and another is reflected back toward the oscillator. Across the junction from left to right, we have the following matching rules (which follows from continuity of the amplitude and its first derivative)

$$\begin{aligned}\cos\left[\omega\frac{(x-L)}{c_l}\right] &\to \cos\left[\omega\frac{(x-L)}{c_r}\right], \\ \sin\left[\omega\frac{(x-L)}{c_l}\right] &\to \left(\frac{c_r}{c_l}\right)\sin\left[\omega\frac{(x-L)}{c_r}\right], \end{aligned} \tag{25}$$

so that the solution for $x\le L$ satisfying the boundary condition at the oscillator

$$\sin[\omega x/c_l+\phi(\omega)] = \sin[\omega(x-L)/c_l]\ \cos[\phi(\omega)+\omega L/c_l]+\cos[\omega(x-L)/c_l]\ \sin[\phi(\omega)+\omega L/c_l], \tag{26}$$

for $x \geq L$ transforms into

$$(c_r/c_l) \cdot \sin[\omega(x-L)/c_r] \ \cos[\phi(\omega) + \omega L/c_l] + \cos[\omega(x-L)/c_r] \ \sin[\phi(\omega) + \omega L/c_l], \quad (27)$$

which has the overall amplitude

$$A = \sqrt{(c_r/c_l)^2 \ \cos^2[\phi(\omega) + \omega L/c_l] + \sin^2[\phi(\omega) + \omega L/c_l]}. \quad (28)$$

Since the string is semi-infinite, it is this amplitude alone that determines the normalization of the modes on the string. The amplitude on the interval $[L, +\infty)$ infinitely outweighs that on $[0, L]$. It follows that the spectral density takes the form

$$\rho(\omega) = \bar{N} \frac{\sin^2[\phi(\omega)]}{(c_r/c_l)^2 \ \cos^2[\phi(\omega) + \omega L/c_l] + \sin^2[\phi(\omega) + \omega L/c_l]} \quad (29)$$

where $\bar{N}$ is a normalization constant. Qualitatively, the density has the form of an ordinary resonance (from the numerator) masked by a function with spikes at $\phi(\omega) + \omega L/c_l = n\pi$ of a certain width. The closer the reflection coefficient $R = (c_l - c_r)/(c_l + c_r)$ is to unity, the sharper the spikes. In the extreme limit $c_r \to 0$, the reflection is total, so that the system is effectively no longer semi-infinite but rather a finite cavity, with a discrete spectrum. In this case the spikes have a δ-function character. The presence of these periodically spaced spikes may be understood as follows. The normalization of the modes of a given ω is determined by the amplitude of the wave at $x > L$. However, it is only near a resonance that this wave has a significant penetration into the region $x < L$ to the left of the junction.

Consider the Fourier transform of

$$f(T) = \sum_{n=-\infty}^{+\infty} R^{|n|} \exp[i\delta n] \exp[-i\bar{\omega}(T - 2nL)] \exp\left[-\frac{\gamma}{2}|T - 2nL/c_l|\right], \quad (30)$$

which represents the effect on the commutator of (multiple) reflections, with the real reflection coefficient R and phase shift δ describing the effect of a single reflection. This Fourier transform is simply

$$\rho(\omega) = \rho_0(\omega) \sum_{n=-\infty}^{+\infty} R^{|n|} \exp[in\delta] \exp[+2in\omega L/c_l] \quad (31)$$

where $\rho_0(\omega)$ is the Fourier transform of $\exp[-i\bar{\omega}T] \exp\Big[-(\gamma/2)|T|\Big]$. We may evaluate

$$\sum_{n=-\infty}^{+\infty} R^{|n|} \exp[in\delta] \exp[+2in\omega L/c_l] = \frac{1 - R^2}{1 + R^2 - R\Big(\exp[+i(2\omega L/c_l + \delta)] + \exp[-i(2\omega L/c_l + \delta)]\Big)} \quad (32)$$

Similarly, using the fact that $\phi(\omega) \approx \pi/2$ in the neighborhood of the resonance, we may rewrite the denominator in eqn. (29) as

$$
\begin{aligned}
&\frac{1}{(c_r/c_l)^2\ \cos^2[\phi(\omega)+\omega L/c_l]+\sin^2[\phi(\omega)+\omega L/c_l]} \approx \frac{1}{(c_r/c_l)^2\ \sin^2[\omega L/c_l]+\cos^2[\omega L/c_l]}\\
&= \frac{2}{(1+c_r^2/c_l^2)+(1-c_r^2/c_l^2)\cos[2\omega L/c_l]}\\
&\propto \frac{1}{1+\lambda\cos[2\omega L/c_l]}
\end{aligned}
\tag{33}
$$

where $\lambda = (c_l^2 - c_r^2)/(c_l^2 + c_r^2) = 2R/(1+R^2)$. We observe that eqn. (33) has the form of eqn. (30) with phase shift $\delta = \pi$.

The above example illustrates how for models having a time translation symmetry nonlocal effects are encoded in the spectral density $\rho(\omega)$. Here the multiple reflections alter the spectral density $\rho_0(\omega)$ in the absence of reflections through multiplication by the mask function in eqn. (33) having the profile of a picket fence. After an integer number of reflection times $2L$ the phases of the various pickets interfere constructively causing a peak in the commutator.

We may also contemplate examples where the linear density of the string $\mu(x)$ varies smoothly so that there is an amplitude for reflection everywhere and these reflections interfere with each other to give a rather complicated response kernel $G(t-t')$. The method of calculating this nonlocal kernel for the brane-bulk system shall be the subject of sections IV and V.

C. Interpretation in terms of Hilbert space

Let us consider the physical interpretation of this diminution in magnitude of the commutator. Our Hilbert space $\mathcal{H}_{string}$ has the structure of a direct product of an infinite number of harmonic oscillator Hilbert spaces. There exists an infinite number of particle species. In an abstract sense, let $f(\omega)$ be a normalized square-integrable complex function, that is

$$\int_0^{+\infty} d\omega\ |f(\omega)|^2 = 1. \tag{34}$$

Each such function specifies a type of particle whose annihilation operator is given by

$$\hat{a}_f = \int_0^{+\infty} d\omega\ f(\omega)\ \hat{a}(\omega), \tag{35}$$

which together with its conjugate $\hat{a}_f^\dagger$ generates a single-particle subspace $\mathcal{H}_f \subset \mathcal{H}_{string}$. We may also consider the Hilbert space of "particle species" (distinct from the above spaces) endowed with a natural inner product:

$$\langle f|g\rangle = \int_0^{+\infty} d\omega\ f^*(\omega)\ g(\omega). \tag{36}$$

More precisely, particle species are labeled by complex rays in this Hilbert space: any two such complex-valued, square-integrable functions differing only by a complex phase represent the same kind of particle.

Without coupling to the string, the type of particle coupled to the oscillator does not evolve with time. $\hat{a}(t) = \exp[-i\bar{\omega}t]\hat{a}(t=0)$, and as noted a difference in phase does not correspond to a difference in particle type. However, in the case of coupling to the string, the oscillator annihilation operator, because it is a superposition of string modes of differing frequency, evolves in this ray space of different particle species. Any quantum operator $\mathcal{O}(t)$ localized at the oscillator at the end of the string at a time t may be expressed as a sum of products of $\hat{a}_{osc}(t)$ and $\hat{a}_{osc}^\dagger(t)$. Any operator that cannot be so constructed is not localized at the end of the string.

Physically, the fact that for large t, $\hat{a}_{osc}(t)$ almost commutes with $\hat{a}_{osc}(t=0)$ reflects the fact that the particles created by $\hat{a}_{osc}^\dagger(t)$ escape from the neighborhood of the end of the string and propagate down the string, eventually leaving no trace detectable by an observer able only to operate on the string end. Once the particle has escaped from the end of the string, no operator localized at the string end is capable of altering its state.

Quantum mechanically, from the point of view of a quantum observer only coupled to the mass at the end of the string, calculating probabilities that involve "dissipation"—that is, the escape of energy, particles, and information down the string—requires summing over all such final states. That is, rather than considering vacuum-to-vacuum amplitudes we must sum over the particles not seen that escape down the string.

Let us first consider the simplest possible process, the creation of a single oscillator quantum at $t = t_i$ and the outcome of an attempt to recover this particle at a subsequent time t_f. One possible outcome is that the particle is detected at $t = t_f$. The amplitude for this process is

$$\mathcal{A} = \langle 0|\hat{a}_{osc}(t_f)\ \hat{a}_{osc}^\dagger(t_i)|0\rangle = \exp[-i\bar{\omega}(t_f - t_i)]\exp[-(\gamma/2)(t_f - t_i)]. \tag{37}$$

Note that the probability

$$p_{1_f \leftarrow 1_i} = |\mathcal{A}|^2 = \exp[-\gamma(t_f - t_i)] \tag{38}$$

is strictly less than one, and

$$p_{0_f \leftarrow 1_i} = 1 - p_{1_f \leftarrow 1_i} = 1 - \exp[-\gamma(t_f - t_i)] \tag{39}$$

gives the probability that no particle is detected at the end of the string. This latter probability is actually the sum of the probabilities for an infinite number of orthogonal processes. One takes the infinite sum over I where the index I labels all the particle types orthogonal to the type coupled to the string at $t = t_f$. In other words,

$$p_{0_f \leftarrow 1_i} = \sum_I |\mathcal{A}_{I_f \leftarrow 1_i}|^2. \tag{40}$$

More complicated 'dissipative' processes may be calculated analogously.

III. EXAMPLES WITHOUT TIME TRANSLATION INVARIANCE

In the previous section we considered harmonic oscillators coupled to a string. Our principal tool was the decomposition into Fourier modes of fixed frequency with respect to time. To quantize we associated the positive frequency modes with the annihilation operators of the unique preferred vacuum state. In this section we generalize to situations where a global time translation symmetry is lacking. We are in particular interested in cases where the coefficients characterizing the coupling of the oscillator or oscillators to the continuum evolve with time. In this case Fourier methods cannot be used. Instead it is necessary to work in real space. In a braneworld cosmology the expansion of the universe on the brane renders the system time dependent.

For the most general case, it is not possible to single out a unique preferred vacuum. However for a massless wave equation, for which any solution may be decomposed into left-moving and right-moving waves, this is not a problem because it suffices to postulate the usual vacuum state for the incoming waves. In other words, where $\omega > 0$ the modes having the form $\exp[-i\omega(t + x)]$ in the limit $t \to -\infty$ are associated with annihilation operators and the $\exp[+i\omega(t+x)]$ modes with the creation operators. This defines a unique

vacuum state, even when the couplings to the harmonic oscillators at the end of the string and of the harmonic oscillators themselves vary with time. The same applies when the equation becomes massless sufficiently quickly as $x \to \infty$. With respect to the oscillators, it is necessary to define an initial quantum state if the oscillators decouple from the string sufficiently quickly as $t \to -\infty$. If they do not decouple rapidly enough, the initial state becomes erased because the initial state of the string degrees of freedom are infinitely more relevant.

In the first worked example we consider a single oscillator whose spring constant, which is proportional to $\omega^2(t)$, varies with time. If $\omega^2(t)$ approaches a constant as $|t|$ becomes large, there are well-defined "in" and "out" vacua. However, these vacua do not coincide with each other. The time variation in $\omega^2(t)$ excites quanta through parametric resonance. The production of quanta may be characterized by a Bogoliubov (i.e., linear canonical) transformation relating the modes of positive and negative frequency for the "in" and the "out" vacua. In this worked example we consider the special case where $\omega^2(t)$ has the form of a step function. In this case, the applicable integrals may be evaluated explicitly. However, the generalization to an arbitrary form for $\omega^2(t)$ approaching a constant for large $|t|$ is straightforward.

The second worked example is analogous except that rather than one there are two harmonic oscillator degrees of freedom at the end of the string. In this example, for $|t| > T$ the two harmonic oscillators are uncoupled. Consequently, to define an initial state, in addition to requiring the absence of incoming waves it is necessary to characterize the ground state of the oscillator initially uncoupled to the string. The resulting Bogoliubov transformation from "in" to "out" states acts on a Hilbert space with one discrete degree of freedom and a continuum of string degrees of freedom. The generalization to many oscillators or to more complicated time dependences is straightforward.

A. A single harmonic oscillator with time varying spring constant

We return to example of section IIA where the stiffness of the oscillator $\bar{\omega}^2$, formerly constant, now becomes a time-dependent function $\bar{\omega}^2(t)$. We consider the simplest system for which nontrivial Bogoliubov coefficients may be calculated explicitly. Concretely, con-

sider a harmonic oscillator of mass m whose coupling to a string of tension τ is turned on for only a finite interval of time. The equation of motion is

$$m\left[\frac{d^2}{dt^2}+\bar{\omega}^2(t)\right]q(t)=F(t) \tag{41}$$

where the driving force $F(t)$ exerted by the string on the oscillator is given by

$$F(t)=\tau\frac{\partial u}{\partial x}\bigg|_{x=0}. \tag{42}$$

We may decompose the string solution, which obeys the massless wave equation, into the two components

$$u(t,x)=u_{out}(t-x)+u_{in}(t+x). \tag{43}$$

The solution for the oscillator is simply

$$q(t)=u_{out}(t)+u_{in}(t). \tag{44}$$

It follows that

$$F(t)=\tau\frac{\partial u}{\partial x}\bigg|_{x=0}=\tau\left(\frac{\partial u_{in}(t)}{\partial t}-\frac{\partial u_{out}(t)}{\partial t}\right). \tag{45}$$

With $\gamma=(\tau/m)$, the equation of motion takes the form

$$\left[\frac{d^2}{dt^2}+\gamma\frac{d}{dt}+\bar{\omega}^2(t)\right]u_{out}(t)=(-)\left[\frac{d^2}{dt^2}-\gamma\frac{d}{dt}+\bar{\omega}^2(t)\right]u_{in}(t) \tag{46}$$

or more symbolically

$$D_{out}(t)\ u_{out}(t)=(-)D_{in}(t)\ u_{in}(t). \tag{47}$$

Formally,

$$u_{out}(t)=(-)\int_{-\infty}^{t}dt'\ G_{out}(t\mid t')\ D_{in}(t')\ u_{in}(t') \tag{48}$$

where G_{out} is the retarded form of D_{out}^{-1}. This means that G_{out} mediates the propagation from t' to t of the state u_{in} subject to the potential D_{in}.

Consider the inverse Fourier transform of u_{out}, which can be interpreted as the description of the state u_{out} in the Heisenberg representation. We obtain

$$u_{out}(\omega_f)\ =\ \frac{-1}{2\pi}\int_{-\infty}^{+\infty}dt_f\ e^{+i\omega_f t_f}\ \int_{-\infty}^{t_f}dt_i\ G_{out}(t_f\mid t_i)\ D_{in}(t_i)\ \int_{-\infty}^{+\infty}d\omega_i\ e^{-i\omega_i t_i}\ u_{in}(\omega_i)$$

$$= \int_{-\infty}^{+\infty} d\omega_i \; S(\omega_f|\omega_i) \; u_{in}(\omega_i) \tag{49}$$

where

$$S(\omega_f|\omega_i) = \frac{-1}{2\pi} \int_{-\infty}^{+\infty} dt_f \; e^{+i\omega_f t_f} \; \int_{-\infty}^{t_f} dt_i \; G_{out}(t_f \mid t_i) \; D_{in}(t_i) \; e^{-i\omega_i t_i} \tag{50}$$

is the corresponding propagator in momentum space describing the amplitude that a quantum with frequency ω_i be later observed to have frequency ω_f. If we associate $e^{-i\omega t}$ with the annihilation operator, then $S(\omega_f|\omega_i)$ describes the probability amplitude that a quantum of frequency ω_i be destroyed and a quantum of frequency ω_f be created. There are no homogeneous solutions for u_{out}. u_{out} is completely determined by u_{in}.

In order to work out a concrete example, we set

$$\bar{\omega}^2(t) = \begin{cases} (\gamma/2)^2 + \Omega^2, & \text{if} \quad |t| > T, \\ (\gamma/2)^2 + \tilde{\Omega}^2, & \text{if} \quad |t| < T. \end{cases} \tag{51}$$

It follows that the homogeneous equation $D_{out}(t) \; u(t) = 0$ has the solutions

$$u(t) = \begin{cases} A \;\; \exp[(-\gamma/2 + i\Omega)t] + B \;\; \exp[(-\gamma/2 - i\Omega)t], & \text{for} \quad |t| > T, \\ C \;\; \exp[(-\gamma/2 + i\tilde{\Omega})t] + D \;\; \exp[(-\gamma/2 - i\tilde{\Omega})t], & \text{for} \quad |t| < T. \end{cases} \tag{52}$$

We adopt a sort of Heisenberg representation where, here for $|t| > T$, the coefficients of the two solutions are expressed with respect to an arbitrary origin at $t = t'$ (or $t = t''$):

$$\begin{aligned} u(t) \;&=\; A_{t'} \; e^{(-\gamma/2+i\Omega)(t-t')} + B_{t'} \; e^{(-\gamma/2-i\Omega)(t-t')} \\ &=\; A_{t''} \; e^{(-\gamma/2+i\Omega)(t-t'')} + B_{t''} \; e^{(-\gamma/2-i\Omega)(t-t'')}. \end{aligned} \tag{53}$$

It follows that a change of origin is effected by the transformation

$$\begin{pmatrix} A \\ B \end{pmatrix}_{t''} \equiv \begin{pmatrix} A_{t''} \\ B_{t''} \end{pmatrix} = \mathbf{P}(t'' \mid t') \begin{pmatrix} A \\ B \end{pmatrix}_{t'} = \begin{pmatrix} e^{(-\gamma/2+i\Omega)(t''-t')} & 0 \\ 0 & e^{(-\gamma/2-i\Omega)(t''-t')} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}_{t'}. \tag{54}$$

Similarly, for $|t| < T$,

$$u(t) \;=\; C_{t'} \; e^{(-\gamma/2+i\tilde{\Omega})(t-t')} + D_{t'} \; e^{(-\gamma/2-i\tilde{\Omega})(t-t')} \tag{55}$$

and a change of origin is effected by the transformation

$$\begin{pmatrix} C \\ D \end{pmatrix}_{t''} = \tilde{\mathbf{P}}(t'' \mid t') \begin{pmatrix} C \\ D \end{pmatrix}_{t'} = \begin{pmatrix} e^{(-\gamma/2+i\tilde{\Omega})(t''-t')} & 0 \\ 0 & e^{(-\gamma/2-i\tilde{\Omega})(t''-t')} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}_{t'}. \tag{56}$$

Matching at a jump in $\bar{\omega}^2$, as $(\tilde{\Omega} \leftarrow \Omega)$, is effected by the transformation:

$$\begin{pmatrix} C \\ D \end{pmatrix}_{t+\epsilon} = \mathbf{J} \begin{pmatrix} A \\ B \end{pmatrix}_{t-\epsilon} = \frac{1}{2\tilde{\Omega}} \begin{pmatrix} (\tilde{\Omega}+\Omega) & (\tilde{\Omega}-\Omega) \\ (\tilde{\Omega}-\Omega) & (\tilde{\Omega}+\Omega) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}_{t-\epsilon}, \tag{57}$$

and similarly, in the other direction, as $(\Omega \leftarrow \tilde{\Omega})$,

$$\begin{pmatrix} A \\ B \end{pmatrix}_{t+\epsilon} = \mathbf{J}^{-1} \begin{pmatrix} C \\ D \end{pmatrix}_{t-\epsilon} = \frac{1}{2\Omega} \begin{pmatrix} (\Omega+\tilde{\Omega}) & (\Omega-\tilde{\Omega}) \\ (\Omega-\tilde{\Omega}) & (\Omega+\tilde{\Omega}) \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}_{t-\epsilon}. \tag{58}$$

In order to construct Green's functions, it is useful to divide the the real line into three regions

$$\begin{aligned} \text{region I} &= (-\infty, -T], \\ \text{region II} &= (-T, +T), \\ \text{region III} &= [+T, +\infty). \end{aligned} \tag{59}$$

A Green's function for the oscillator where both endpoints lie within region I (i.e., $t_i < t_f < -T$), is expressed in terms of the conventional notation as

$$G(t, t_i) = (\partial_t^2 + \gamma\partial_t + \omega^2)^{-1}(t, t_i) = \frac{\sin[\Omega(t-t_i)]}{\Omega} \exp[-(\gamma/2)(t-t_i)]\ \theta(t-t_i). \tag{60}$$

In terms of our notation, where the homogeneous solutions are expressed according to the conventions of eqns. (53) and (54), the Green's function is generated by the column vector

$$\begin{pmatrix} \frac{+1}{2i\Omega} \\ \frac{-1}{2i\Omega} \end{pmatrix} \tag{61}$$

localized at $t = t_i$ and placed at the far right in matrix expressions time-ordered from right to left. It follows that, in terms of our matrix notation, the Green's function is re-expressed as

$$G(t_f\ |t_i) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \mathbf{P}(t_f \mid t_i)\ \begin{pmatrix} \frac{+1}{2i\Omega} \\ \frac{-1}{2i\Omega} \end{pmatrix}. \tag{62}$$

For propagation from region I into region II one instead would have

$$G(t_f\ |t_i) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \tilde{\mathbf{P}}(t_f \mid -T)\ \mathbf{J}\ \mathbf{P}(-T\ |t_i)\ \begin{pmatrix} \frac{+1}{2i\Omega} \\ \frac{-1}{2i\Omega} \end{pmatrix}. \tag{63}$$

and the other cases may be worked out analogously in a straightforward manner. If t_i lies in regions I or III, the column vector (61) is used; otherwise, if t_i lies in region II, Ω in eqn. (61) is replaced with $\tilde{\Omega}$.

We now proceed to express the S matrix as the following sum

$$\begin{aligned} S(\omega_f|\omega_i) &= S_{I\leftarrow I}(\omega_f|\omega_i) + S_{II\leftarrow I}(\omega_f|\omega_i) + S_{III\leftarrow I}(\omega_f|\omega_i) \\ &+ S_{II\leftarrow II}(\omega_f|\omega_i) + S_{III\leftarrow II}(\omega_f|\omega_i) + S_{III\leftarrow III}(\omega_f|\omega_i), \end{aligned} \tag{64}$$

whose six terms are given by:

$$\begin{aligned} S_{I\leftarrow I}(\omega_f|\omega_i) &= \frac{1}{2\pi}(-)\left[\frac{\gamma^2}{4} + \Omega^2 - \omega_i^2 + i\gamma\omega_i\right]\int_{-\infty}^{-T} dt_f \int_{-\infty}^{t_f} dt_i\ e^{+i\omega_f t_f} e^{-i\omega_i t_i} \\ &\times \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \mathbf{P}(t_f|\ t_i) \begin{pmatrix} \frac{+1}{2i\Omega} \\ \frac{-1}{2i\Omega} \end{pmatrix}. \end{aligned} \tag{65}$$

$$\begin{aligned} S_{II\leftarrow I}(\omega_f|\omega_i) &= \frac{1}{2\pi}(-)\left[\frac{\gamma^2}{4} + \Omega^2 - \omega_i^2 + i\gamma\omega_i\right]\int_{-T}^{+T} dt_f \int_{-\infty}^{-T} dt_i\ e^{+i\omega_f t_f} e^{-i\omega_i t_i} \\ &\times \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \tilde{\mathbf{P}}(t_f|-T)\ \mathbf{J}\ \mathbf{P}(-T|\ t_i) \begin{pmatrix} \frac{+1}{2i\Omega} \\ \frac{-1}{2i\Omega} \end{pmatrix}. \end{aligned} \tag{66}$$

$$\begin{aligned} S_{III\leftarrow I}(\omega_f|\omega_i) &= \frac{1}{2\pi}(-)\left[\frac{\gamma^2}{4} + \Omega^2 - \omega_i^2 + i\gamma\omega_i\right]\int_{+T}^{+\infty} dt_f \int_{-\infty}^{-T} dt_i\ e^{+i\omega_f t_f} e^{-i\omega_i t_i} \\ &\times \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \mathbf{P}(t_f|+T)\ \mathbf{J}^{-1}\ \tilde{\mathbf{P}}(+T\ |-T)\ \mathbf{J}\ \mathbf{P}(-T\ |\ t_i) \begin{pmatrix} \frac{+1}{2i\Omega} \\ \frac{-1}{2i\Omega} \end{pmatrix}. \end{aligned} \tag{67}$$

$$\begin{aligned} S_{II\leftarrow II}(\omega_f|\omega_i) &= \frac{1}{2\pi}(-)\left[\frac{\gamma^2}{4} + \tilde{\Omega}^2 - \omega_i^2 + i\gamma\omega_i\right]\int_{-T}^{+T} dt_f \int_{-T}^{t_f} dt_i\ e^{+i\omega_f t_f} e^{-i\omega_i t_i} \\ &\times \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \tilde{\mathbf{P}}(t_f\ |\ t_i) \begin{pmatrix} \frac{+1}{2i\tilde{\Omega}} \\ \frac{-1}{2i\tilde{\Omega}} \end{pmatrix}. \end{aligned} \tag{68}$$

$$\begin{aligned} S_{III\leftarrow II}(\omega_f|\omega_i) &= \frac{1}{2\pi}(-)\left[\frac{\gamma^2}{4} + \tilde{\Omega}^2 - \omega_i^2 + i\gamma\omega_i\right]\int_{+T}^{+\infty} dt_f \int_{-T}^{+T} dt_i\ e^{+i\omega_f t_f} e^{-i\omega_i t_i} \\ &\times \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \tilde{\mathbf{P}}(t_f\ |+T)\ \mathbf{J}^{-1}\ \mathbf{P}(+T\ |\ t_i)\ \begin{pmatrix} \frac{+1}{2i\tilde{\Omega}} \\ \frac{-1}{2i\tilde{\Omega}} \end{pmatrix}. \end{aligned} \tag{69}$$

$$\begin{aligned} S_{III\leftarrow III}(\omega_f|\omega_i) &= \frac{1}{2\pi}(-)\left[\frac{\gamma^2}{4} + \Omega^2 - \omega_i^2 + i\gamma\omega_i\right]\int_{+T}^{+\infty} dt_f \int_{+T}^{t_f} dt_i\ e^{+i\omega_f t_f} e^{-i\omega_i t_i} \\ &\times \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T \mathbf{P}(t_f\ |\ t_i) \begin{pmatrix} \frac{+1}{2i\Omega} \\ \frac{-1}{2i\Omega} \end{pmatrix}. \end{aligned} \tag{70}$$

We next proceed to calculate these six terms more explicitly. First, for $S_{I\leftarrow I}$ we may extract and evaluate the integral

$$\begin{aligned}
&\int_{-\infty}^{-T} dt_f \int_{-\infty}^{t_f} dt_i \; e^{+i\omega_f t_f} \; e^{-i\omega_i t_i} \; \begin{pmatrix} e^{(-\gamma/2+i\Omega)(t_f-t_i)} & 0 \\ 0 & e^{(-\gamma/2-i\Omega)(t_f-t_i)} \end{pmatrix} \\
&= \int_{-\infty}^{-T} dt_f \; e^{+i(\omega_f-\omega_i)t_f} \int_0^{+\infty} d\tau \; e^{+i\omega_i\tau} \begin{pmatrix} e^{(-\gamma/2+i\Omega)\tau} & 0 \\ 0 & e^{(-\gamma/2-i\Omega)\tau} \end{pmatrix} \\
&= e^{-i(\omega_f-\omega_i)T} \; \frac{(-i)}{(\omega_f-\omega_i)-i\epsilon} \begin{pmatrix} \frac{1}{(\gamma/2-i\Omega-i\omega_i)} & 0 \\ 0 & \frac{1}{(\gamma/2+i\Omega-i\omega_i)} \end{pmatrix}.
\end{aligned} \tag{71}$$

For $S_{II\leftarrow I}$, we may evaluate the two integrals:

$$\int_{-\infty}^{-T} dt_i \; e^{-i\omega_i t_i} \; e^{(-\gamma/2\pm i\Omega)(-T-t_i)} = e^{+i\omega_i T} \frac{1}{\gamma/2 \mp i\Omega - i\omega_i} \tag{72}$$

and

$$\int_{-T}^{+T} dt_f \; e^{+i\omega_f t_f} \; e^{(-\gamma/2\pm i\tilde{\Omega})(t_f+T)} = \frac{e^{-i\omega_f T} - e^{+i\omega_f T} e^{(-\gamma/2\pm i\tilde{\Omega})2T}}{\gamma/2 \mp i\tilde{\Omega} - i\omega_f}. \tag{73}$$

For $S_{III\leftarrow I}$, we have the first integral from $S_{II\leftarrow I}$ again and also

$$\int_{+T}^{+\infty} dt_f \; e^{+i\omega_f t_f} \; e^{(-\gamma/2\pm i\Omega)(t_f-T)} = e^{+i\omega_f T} \frac{1}{\gamma/2 \mp i\Omega - i\omega_f}. \tag{74}$$

$S_{II\leftarrow II}$ has the two nested integrals

$$\begin{aligned}
&\int_{-T}^{+T} dt_f \int_{-T}^{t_f} dt_i \; e^{+i\omega_f t_f} \; e^{-i\omega_i t_i} \; e^{(-\gamma/2\pm i\tilde{\Omega})(t_f-t_i)} \\
&= \int_{-T}^{+T} dt_f \; e^{(-\gamma/2\pm i\tilde{\Omega}+i\omega_f)t_f} \int_{-T}^{t_f} dt_i \; e^{(+\gamma/2\mp i\tilde{\Omega}-i\omega_i)t_i} \\
&= \int_{-T}^{+T} dt_f \; e^{(-\gamma/2\pm i\tilde{\Omega}+i\omega_f)t_f} \; \frac{e^{(\gamma/2\mp i\tilde{\Omega}-i\omega_i)t_f} - e^{-(\gamma/2\mp i\tilde{\Omega}-i\omega_i)T}}{\gamma/2 \mp i\tilde{\Omega} - i\omega_i} \\
&= \frac{1}{\gamma/2 \mp i\tilde{\Omega} - i\omega_i} \left[\frac{e^{+i(\omega_f-\omega_i)T} - e^{-i(\omega_f-\omega_i)T}}{i(\omega_f-\omega_i)} + \frac{e^{[2(-\gamma/2\pm i\tilde{\Omega})+i(\omega_i+\omega_f)]T} - e^{+i(\omega_i-\omega_f)T}}{\gamma/2 \mp i\tilde{\Omega} - i\omega_f} \right]
\end{aligned} \tag{75}$$

$S_{III\leftarrow II}$ has the integral

$$\int_{-T}^{+T} dt_i \; e^{-i\omega_i t_i} \; e^{(-\gamma/2\pm i\tilde{\Omega})(-t_i+T)} = \frac{e^{-i\omega_i T} - e^{+i\omega_i T} e^{(-\gamma/2\pm i\tilde{\Omega})2T}}{\gamma/2 \mp i\tilde{\Omega} - i\omega_i}. \tag{76}$$

$S_{III\leftarrow III}$ has the integral

$$\begin{aligned}
&\int_{+T}^{+\infty} dt_f \int_{+T}^{t_f} dt_i \; e^{+i\omega_f t_f} \; e^{-i\omega_i t_i} \; e^{(-\gamma/2\pm i\Omega)(t_f-t_i)} \\
&= \int_{+T}^{+\infty} dt_f \; e^{+i(\omega_f-\omega_i)t_f} \int_0^{t_f-T} d\tau \; e^{+i\omega_i\tau} e^{(-\gamma/2\pm i\Omega)\tau}
\end{aligned}$$

$$= \frac{e^{+i(\omega_f-\omega_i)T}}{\gamma/2 \mp i\Omega - i\omega_i}\left[\frac{i}{(\omega_f-\omega_i)+i\epsilon} - \frac{1}{\gamma/2 \mp i\Omega - i\omega_f}\right]. \tag{77}$$

We have used the fact that

$$\int_0^{+\infty} dt\; e^{+i\omega t} = \frac{+i}{\omega+i\epsilon}, \quad \int_{-\infty}^{0} dt\; e^{+i\omega t} = \frac{-i}{\omega-i\epsilon}. \tag{78}$$

For further reference, we note that

$$2\pi\delta(\omega) = (+i)\left(\frac{1}{\omega+i\epsilon} - \frac{1}{\omega-i\epsilon}\right), \qquad P\left(\frac{1}{\omega}\right) = \frac{1}{2}\left[\frac{1}{\omega+i\epsilon} + \frac{1}{\omega-i\epsilon}\right] \tag{79}$$

where P denotes principal part.

The final S matrix takes the form

$$\begin{aligned} S(\omega_f|\omega_i) \;&=\; -2\delta(\omega_f-\omega_i)\,\frac{\omega_f^2-\Omega^2-\gamma^2/4-i\gamma\omega_f}{\omega_f^2-\Omega^2-\gamma^2/4+i\gamma\omega_f} \\ &+\; S_{nonsing}(\omega_f|\omega_i) \end{aligned} \tag{80}$$

where the singular δ-function term reflects what S would be if $\Omega = \tilde{\Omega}$ (i.e., if the string constant of the oscillator were unchanged) and $S_{nonsing}$ is the smooth part resulting from the Fourier transform of the wave packet emitted as a result of the change during the interval $[-T,+T]$.

The non-singular contribution to the final S matrix is given by

$$\begin{aligned} &- \left[\left(\frac{\omega_i^2-\tilde{\Omega}^2-\gamma^2/4-i\gamma\omega_i}{\omega_i^2-\tilde{\Omega}^2-\gamma^2/4+i\gamma\omega_i}\right) - \left(\frac{\omega_i^2-\Omega^2-\gamma^2/4-i\gamma\omega_i}{\omega_i^2-\Omega^2-\gamma^2/4+i\gamma\omega_i}\right)\right]\frac{\sin[(\omega_f-\omega_i)T]}{\pi(\omega_f-\omega_i)} \\ &+ \frac{1}{\pi}\frac{\gamma\omega_i(\Omega^2-\tilde{\Omega}^2)}{\left\{(\omega_i+i\gamma/2)^2-\Omega^2\right\}\left\{(\omega_i+i\gamma/2)^2-\tilde{\Omega}^2\right\}\left\{(\omega_f+i\gamma/2)^2-\Omega^2\right\}\left\{(\omega_f+i\gamma/2)^2-\tilde{\Omega}^2\right\}} \\ &\times\Bigg[\left(\omega_f+\omega_i+i\gamma\right)\times\Bigg\{-\left[(\omega_f+i\gamma/2)^2-\Omega^2\right]\exp[+i(\omega_f-\omega_i)(-T)] \\ &\qquad\qquad + \left[(\omega_f+i\gamma/2)^2-\tilde{\Omega}^2\right]\exp[+i(\omega_f-\omega_i)(+T)]\Bigg\} \\ &\quad + \exp[-\gamma T]\;\exp[+i\omega_f(+T)]\;\exp[-i\omega_i(-T)]\;(\tilde{\Omega}^2-\Omega^2) \\ &\quad\times\left\{(\omega_f+\omega_i+i\gamma)\cos[\tilde{\Omega}(2T)] - i\left[(\omega_i+i\gamma/2)(\omega_f+i\gamma/2)+\tilde{\Omega}^2\right]\frac{\sin[\tilde{\Omega}(2T)]}{\tilde{\Omega}}\right\}\Bigg]. \end{aligned} \tag{81}$$

The term on the first line arises from the fact that the reflection from the oscillator during the interval $[-T,+T]$ occurs with a different phase from that during its complement when $|t| > T$. This term, which approaches a delta function as $T \to \infty$, assumes that during the interval $[-T,+T]$ the oscillator is described by its asymptotic amplitude and phase (as if the

change from Ω to $\tilde{\Omega}$ had taken place in the infinite past and there were no transients). This picture, of course, is an approximation because there were transients, and the remaining terms give the form of these transients. The two terms on the third and fourth lines have the form of scatterers localized in time at $t = -T$ and $t = +T$, respectively. The first is due to the transient as the oscillator changes its amplitude and phase just after $t = -T$ as a result of the change from Ω to $\tilde{\Omega}$. The second term is the same at $t = +T$ as the oscillator frequency changes back from $\tilde{\Omega}$ to Ω under the assumption that the previous initial transient has completely decayed away. Finally, the terms on the last two lines result from the fact that the first transient from $t = -T$ has not completely decayed away by the instant $t = +T$.

The transformations just calculated may be cast into a more general framework. What we have just been doing was calculating the matrix elements of a linear symplectic transformation relating the description of the system in terms of "in" modes to an equivalent description in terms of "out" modes. The two descriptions are related by a linear transformation, which we may express more abstractly as

$$\begin{pmatrix} \hat{a}_{out}(\omega) \\ \hat{a}^{\dagger}_{out}(\omega) \end{pmatrix} = \int_0^{\infty} d\omega' \begin{pmatrix} S_{++}(\omega|\omega') & S_{+-}(\omega|\omega') \\ S_{-+}(\omega|\omega') & S_{--}(\omega|\omega') \end{pmatrix} \begin{pmatrix} \hat{a}_{in}(\omega') \\ \hat{a}^{\dagger}_{in}(\omega') \end{pmatrix}. \tag{82}$$

We know that both the "in" and the "out" operators must satisfy the canonical commutation relations. In particular, for the "in" operators, we have

$$\begin{aligned} [\hat{a}_{in}(\omega), \hat{a}^{\dagger}_{in}(\omega')] &= \delta(\omega - \omega'), \\ [\hat{a}_{in}(\omega), \hat{a}_{in}(\omega')] &= 0, \end{aligned} \tag{83}$$

where $\omega, \omega' \geq 0$, and similarly for the "out" operators

$$\begin{aligned} [\hat{a}_{out}(\omega), \hat{a}^{\dagger}_{out}(\omega')] &= \delta(\omega - \omega'), \\ [\hat{a}_{out}(\omega), \hat{a}_{out}(\omega')] &= 0. \end{aligned} \tag{84}$$

However, the linear transformation (82) allows us to calculate the "out" commutation relation using eqn. (83). The requirement that the commutation relations thus obtained agree with those in (84) leads to a number of consistency conditions for the coefficients in (82), which we give below.

By taking Hermitian conjugates, we obtain that $S_{\alpha\beta}(\omega|\omega')$ satisfies the conditions:

$$S_{++}(\omega|\omega') = S^{*}_{--}(\omega|\omega'),$$

$$S_{+-}(\omega|\omega') \;=\; S^*_{-+}(\omega|\omega'). \tag{85}$$

Eqn. (84) gives

$$\int_0^\infty d\omega'' \;\left[S_{++}(\omega|\omega'')S^*_{++}(\omega'|\omega'') - S_{+-}(\omega|\omega'')S^*_{+-}(\omega'|\omega'')\right] = \delta(\omega-\omega') \tag{86}$$

and

$$\int_0^\infty d\omega'' \;[S_{++}(\omega|\omega'')S_{+-}(\omega'|\omega'') - S_{+-}(\omega|\omega'')S_{++}(\omega'|\omega'')] = 0. \tag{87}$$

These conditions are necessary and sufficient.

For the "out" states or quanta, we may construct a number density operator $\hat{N}_{out}(\omega) = \hat{a}^\dagger_{out}(\omega)\hat{a}_{out}(\omega)$ indicating the density of the number of particles of frequency ω in the "out" space. We find that the expectation value of $\hat{N}_{out}(\omega)$ for the "in" vacuum state is

$$\langle 0_{in}|\hat{N}_{out}(\omega)|0_{in}\rangle = \int_0^\infty d\omega'|S_{+-}(\omega|\omega')|^2. \tag{88}$$

It follows that the "in" and "out" vacua will be different as long as

$$\langle N_{out}\rangle = \int_0^\infty d\omega\;\langle N_{out}(\omega)\rangle = \int_0^\infty d\omega\;\int_0^\infty d\omega'|S_{+-}(\omega|\omega')|^2 > 0, \tag{89}$$

that is, as long as the expectation value of the operator $\hat{N}_{out} = \int d\omega\hat{N}_{out}(\omega)$ for the total number of "out" particles created from the "in" vacuum state is different from zero. We will obtain a vanishing expectation value, hence coinciding vacua, in the case when the string-oscillator coupling remains constant.

B. A pair of harmonic oscillators with time dependent parameters

In this section, generalizing the techniques of the example, we solve the equations for a pair of coupled harmonic oscillators with a forcing term included

$$m\left[\frac{d^2}{dt^2} + \begin{pmatrix}\omega_1^2(t) & \alpha^2(t)\\ \alpha^2(t) & \omega_2^2(t)\end{pmatrix}\right]\begin{pmatrix}q_1(t)\\ q_2(t)\end{pmatrix} = \begin{pmatrix}F_1(t)\\ F_2(t)\end{pmatrix}. \tag{90}$$

Here $\alpha^2(t)$ is the coupling between the oscillators, which we shall make approach to zero sufficiently fast as $|t| \to 0$. We attach only oscillator 2 to the string so that

$$q_1(t) = q_{osc}(t), \quad q_2(t) = u_{out}(t) + u_{in}(t) \tag{91}$$

where the solution on the string is

$$u(t,x) = u_{out}(t-x) + u_{in}(t+x) \tag{92}$$

for $x \geq 0$. It follows that

$$F_1(t) = 0, \quad F_2(t) = \tau \frac{\partial u}{\partial x}\bigg|_{x=0} = \tau \left(\frac{\partial u_{in}(t)}{\partial t} - \frac{\partial u_{out}(t)}{\partial t} \right). \tag{93}$$

The equation of motion takes the form

$$\begin{aligned} &\left[\frac{d^2}{dt^2} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix} \frac{d}{dt} + \begin{pmatrix} \omega_1^2 & \alpha^2(t) \\ \alpha^2(t) & \omega_2^2 \end{pmatrix} \right] \begin{pmatrix} q_{osc}(t) \\ u_{out}(t) \end{pmatrix} \\ &= \begin{pmatrix} D^1(t) \\ D^2(t) \end{pmatrix} u_{in}(t) = \begin{pmatrix} -\alpha^2(t) \\ -\frac{d^2}{dt^2} + \gamma \frac{d}{dt} - \omega_2^2 \end{pmatrix} u_{in}(t), \end{aligned} \tag{94}$$

where we set

$$\alpha^2(t) = \begin{cases} \bar{\alpha}^2, & \text{if} \quad |t| < T, \\ 0, & \text{if} \quad |t| > T. \end{cases} \tag{95}$$

For $|t| > T$, we may write the homogeneous solution in the form

$$E^i_{(a)} \ e^{-\gamma_{(a)} t} \tag{96}$$

where

$$\begin{aligned} +i\gamma_{(1)} &= \omega_{(1)} = +\omega_1, \\ +i\gamma_{(2)} &= \omega_{(2)} = -\omega_1, \\ +i\gamma_{(3)} &= \omega_{(3)} = +i[\gamma/2 + \sqrt{(\gamma/2)^2 - \omega_2^2}], \\ +i\gamma_{(4)} &= \omega_{(4)} = +i[\gamma/2 - \sqrt{(\gamma/2)^2 - \omega_2^2}] \end{aligned} \tag{97}$$

and

$$E^i_{(1)} = E^i_{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad E^i_{(3)} = E^i_{(4)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{98}$$

For $|t| < T$, the $\tilde{\omega}_{(\tilde{a})}$'s are the roots ω of

$$\begin{vmatrix} \omega^2 - \omega_1^2 & -\bar{\alpha}^2 \\ -\bar{\alpha}^2 & \omega^2 - \omega_2^2 + i\gamma\omega \end{vmatrix} = 0, \tag{99}$$

or equivalently of the quartic polynomial

$$(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2 + i\gamma\omega) - \bar{\alpha}^4 = 0. \tag{100}$$

[Note that for $\gamma \neq$, $\bar{\alpha} \neq 0$ this quartic does not reduce to a quadratic in ω^2.] The corresponding eigenvectors are given by $\tilde{E}^i_{(\tilde{a})}$, so that

$$\tilde{E}^i_{(\tilde{a})}\; e^{-\tilde{\gamma}_{(\tilde{a})} t} \tag{101}$$

form a complete basis for the homogeneous solutions. The indices $(\tilde{a})$ refer to the basis for $|t| < T$ whereas the indices (a) refer to the basis for $|t| > T$.

We define the matching matrix $J_{(\tilde{a})(a)}$ so that for all (a) and i

$$\begin{aligned} \tilde{E}^i_{(\tilde{a})} &= \sum_{(\tilde{a})} J_{(\tilde{a})(a)}\; E^i_{(a)}, \\ \tilde{\gamma}_{(\tilde{a})} \tilde{E}^i_{(\tilde{a})} &= \sum_{(\tilde{a})} J_{(\tilde{a})(a)}\; \gamma_{(a)} E^i_{(a)}. \end{aligned} \tag{102}$$

We define $G^i_{(a)}$ such that for all $i,\ j$

$$\begin{aligned} &\sum_{(a)} G^i_{(a)}\; E^j_{(a)} = 0, \\ &\sum_{(a)} \gamma_{(a)}\; G^i_{(a)}\; E^j_{(a)} = \delta_{ij}, \end{aligned} \tag{103}$$

and the $\tilde{G}^i_{(a)}$ are analogously defined.

Consequently, for t, t' entirely in region I (or entirely in region III) the Green's function is given by

$$G^{ij}(t,t') = E^i_{(a)}\; P_{(a)(b)}(t,t')\; G^j_{(b)}, \tag{104}$$

where

$$P_{(a)(b)}(t,t') = \delta_{(a),(b)}\; e^{-\gamma_{(a)}(t-t')} = P_{(a)}(t,t')\; \delta_{(a),(b)} \tag{105}$$

describes a change of origin, and similarly for points entirely in region II,

$$G^{ij}(t,t') = \tilde{E}^i_{(\tilde{a})}\; \tilde{P}_{(\tilde{a})(\tilde{b})}(t,t')\; G^j_{(\tilde{b})}. \tag{106}$$

When points straddle the regions, example for propagation from I into II one obtains

$$G^{ij}(t,t') = \tilde{E}^i_{(\tilde{a})}\; \tilde{P}_{(\tilde{a})(\tilde{b})}(t,-T)\; J_{(\tilde{b})(b)}\; P_{(b)(a)}(-T,t')\; G^j_{(a)}. \tag{107}$$

We now give the expressions for the various Bogoliubov coefficients.

$$S(\mathrm{osc},+,\mathrm{out} \leftarrow \mathrm{osc},+,\mathrm{in}) = (J^{-1})_{(a=2)(\tilde{a})}\; \tilde{P}_{(\tilde{a})}(+T,-T)\; J_{(\tilde{a})(b=2)} \tag{108}$$

$$S(\text{osc},-,\text{out}\leftarrow\text{osc},+,\text{in})=(J^{-1})_{(a=1)(\tilde a)}\ \tilde P_{(\tilde a)}(+T,-T)\ J_{(\tilde a)(b=2)} \tag{109}$$

$$S(\text{osc},+,\text{out}\leftarrow\text{osc},-,\text{in})=(J^{-1})_{(a=2)(\tilde a)}\ \tilde P_{(\tilde a)}(+T,-T)\ J_{(\tilde a)(b=1)} \tag{110}$$

$$S(\text{osc},-,\text{out}\leftarrow\text{osc},-,\text{in})=(J^{-1})_{(a=1)(\tilde a)}\ \tilde P_{(\tilde a)}(+T,-T)\ J_{(\tilde a)(b=1)} \tag{111}$$

$$\begin{aligned}
&S(\text{osc},+,\text{out}\leftarrow\omega_I,\text{in})=\\
&\quad\int_{-\infty}^{-T}dt_I\ (J^{-1})_{(a=2)(\tilde a)}\ \tilde P_{(\tilde a)}(T,-T)\ J_{(\tilde a)(b)}\ P_{(b)}(-T,t_I)\ G^j_{(b)}D^j(t_I)\ e^{-i\omega_I t_I}\\
&\quad+\int_{-T}^{+T}dt_I\ (J^{-1})_{(a=2)(\tilde a)}\ \tilde P_{(\tilde a)}(+T,t_I)\ \tilde G^j_{(\tilde a)}D^j(t_I)\ e^{-i\omega_I t_I}
\end{aligned} \tag{112}$$

$$\begin{aligned}
&S(\text{osc},-,\text{out}\leftarrow\omega_I,\text{in})=\\
&\quad\int_{-\infty}^{-T}dt_I\ (J^{-1})_{(a=1)(\tilde a)}\ \tilde P_{(\tilde a)}(+T,-T)\ J_{(\tilde a)(b)}\ P_{(b)}(-T,t_I)\ G^j_{(b)}D^j(t_I)\ e^{-i\omega_I t_I}\\
&\quad+\int_{-T}^{+T}dt_I\ (J^{-1})_{(a=1)(\tilde a)}\ \tilde P_{(\tilde a)}(+T,t_I)\ \tilde G^j_{(\tilde a)}D^j(t_I)\ e^{-i\omega_I t_I}
\end{aligned} \tag{113}$$

$$\begin{aligned}
&S(\omega_F,\text{out}\leftarrow\omega_I,\text{in})=\\
&\int_{+T}^{+\infty}dt_F\ e^{+i\omega_F t_F}\ \int_{-\infty}^{-T}dt_I\ E^{i=2}_{(a)}\ P_{(a)}(t_F,+T)\ (J^{-1})_{(a)(\tilde a)}\ \tilde P_{(\tilde a)}(+T,-T)\\
&\quad\times J_{(\tilde a)(b)}\ P_{(b)}(-T,t_I)\ G^j_{(b)}D^j(t_I)\ e^{-i\omega_I t_I}\\
&+\int_{+T}^{+\infty}dt_F\ e^{+i\omega_F t_F}\ \int_{-T}^{+T}dt_I\ E^{i=2}_{(a)}\ P_{(a)}(t_F,+T)\ (J^{-1})_{(a)(\tilde a)}\ \tilde P_{(\tilde a)}(+T,t_I)\ \tilde G^j_{(\tilde a)}D^j(t_I)\ e^{-i\omega_I t_I}\\
&+\int_{+T}^{+\infty}dt_F\ e^{+i\omega_F t_F}\ \int_{+T}^{+\infty}dt_I\ E^{i=2}_{(a)}\ P_{(a)}(t_F,+T)\ G^j_{(a)}D^j(t_I)\ e^{-i\omega_I t_I}\\
&+\int_{-T}^{+T}dt_F\ e^{+i\omega_F t_F}\ \int_{-\infty}^{-T}dt_I\ \tilde E^{i=2}_{(\tilde a)}\ \tilde P_{(\tilde a)}(t_F,-T)\ J_{(\tilde a)(b)}\ P_{(b)}(-T,t_I)\ G^j_{(b)}D^j(t_I)\ e^{-i\omega_I t_I}\\
&+\int_{-T}^{+T}dt_F\ e^{+i\omega_F t_F}\ \int_{-T}^{+T}dt_I\ \tilde E^{i=2}_{(\tilde a)}\ \tilde P_{(\tilde a)}(t_F,t_I)\ \tilde G^j_{(\tilde a)}D^j(t_I)\ e^{-i\omega_I t_I}\\
&+\int_{-\infty}^{-T}dt_F\ e^{+i\omega_F t_F}\ \int_{-\infty}^{-T}dt_I\ E^{i=2}_{(a)}\ P_{(a)}(t_F,t_I)\ G^j_{(a)}D^j(t_I)\ e^{-i\omega_I t_I}
\end{aligned} \tag{114}$$

$$\begin{aligned}
&S(\omega_F,\text{out}\leftarrow\text{osc},+,\text{in})=\\
&\quad\int_{+T}^{+\infty}dt_F\ e^{+i\omega_F t_F}\ E^{i=2}_{(a)}\ P_{(a)}(t_F,+T)\ (J^{-1})_{(a)(\tilde a)}\ \tilde P_{(\tilde a)}(+T,-T)\ J_{(\tilde a)(b=2)}\\
&\quad+\int_{-T}^{+T}dt_F\ e^{+i\omega_F t_F}\ \tilde E^{i=2}_{(\tilde a)}\ \tilde P_{(\tilde a)}(t_F,-T)\ J_{(\tilde a)(b=2)}
\end{aligned} \tag{115}$$

$$\begin{aligned}
&S(\omega_F,\text{out}\leftarrow\text{osc},-,\text{in})=\\
&\quad\int_{+T}^{+\infty}dt_F\ e^{+i\omega_F t_F}\ E^{i=2}_{(a)}\ P_{(a)}(t_F,+T)\ (J^{-1})_{(a)(\tilde a)}\ \tilde P_{(\tilde a)}(+T,-T)\ J_{(\tilde a)(b=1)}
\end{aligned}$$

$$+\int_{-T}^{+T} dt_F \ e^{+i\omega_F t_F} \ \tilde{E}^{i=2}_{(\tilde{a})} \ \tilde{P}_{(\tilde{a})}(t_F, -T) \ J_{(\tilde{a})(b=1)}. \tag{116}$$

For the string mode, a positive frequency for ω_I, ω_F corresponds to the annihilation operators, a negative frequency corresponds to creation operators. In the above formulae, summation over repeated indices is implied, and in some cases indices are repeated three times because intervening Kronecker deltas have been suppressed.

The Bogoliubov coefficients connecting the oscillator to itself, as given by eqns.(108)–(111), can be written as

$$\begin{aligned} S(\text{osc}, S_{out} \leftarrow \text{osc}, S_{in}) \ = \ & [\delta_{S_{out},+}\delta_{2,(a)} + \delta_{S_{out},-}\delta_{1,(a)}] \ [\delta_{S_{in},+}\delta_{2,(b)} + \delta_{S_{in},-}\delta_{1,(b)}] \\ & (J^{-1})_{(a)(\tilde{c})} \ e^{-\gamma_{(\tilde{c})}(2T)} \ J_{(\tilde{c})(b)} \end{aligned} \tag{117}$$

where $S_{in}, S_{out} = +, -$. For large T, the sum over $(\tilde{c})$ is dominated by the mode that decays most slowly among the eigenvalues of eqn. (100), which are in general non-degenerate. We shall denote this mode as $(\tilde{c}_D)$ and its eigenvalue as Γ, so that in this case eqn. (117) above may be approximated as

$$[\delta_{S_{out},+}\delta_{2,(a)} + \delta_{S_{out},-}\delta_{1,(a)}] \ [\delta_{S_{in},+}\delta_{2,(b)} + \delta_{S_{in},-}\delta_{1,(b)}] \ (J^{-1})_{(a)(\tilde{c}_D)} \ J_{(\tilde{c}_D)(b)} \ e^{-\Gamma(2T)}, \tag{118}$$

which approaches zero exponentially fast for large T. This attenuation can be interpreted as the decay of the oscillator excitations into the bulk. When $(\Gamma T) \gg 1$, any initial quantum information about the initial state of the oscillator becomes almost completely dissipated into the bulk. An initial state expressible as a tensor product of oscillator and bulk states becomes highly entangled. This means that if today we measure the state of the oscillator by recourse to operators acting only on the oscillator in terms of the initial state, we are almost exclusively measuring the correlations of the bulk modes—that linear combination that subsequently scatters and excites the oscillator. If the measurement is instead performed just before decoupling, the above holds with the straightforward modification that the second J matrix is suppressed.

Oscillator excitations caused by incoming bulk excitations are described by the Bogoliubov coefficients in eqns. (112)–(113). We can rewrite these coefficients as

$$\begin{aligned} & S(\text{osc}, S_{out} \leftarrow \omega_I, \text{in}) \\ = \ & [\delta_{S_{out},+}\delta_{2,(a)} + \delta_{S_{out},-}\delta_{1,(a)}] \ (J^{-1})_{(a)(\tilde{c})} \ e^{-\gamma_{(\tilde{c})}(2T)} \ J_{(\tilde{c})(b)} \int_{-\infty}^{-T} dt_I \ e^{-\gamma_{(b)}(-T-t_I)} \ G^{j=2}_{(b)} D^{j=2}(t_I) \ e^{-i\omega_I t_I} \end{aligned}$$

$$+\ [\delta_{S_{out},+}\delta_{2,(a)} + \delta_{S_{out},-}\delta_{1,(a)}]\ (J^{-1})_{(a)(\tilde{c})} \int_{-T}^{+T} dt_I\ e^{-\gamma_{(\tilde{c})}(T-t_I)}\ \tilde{G}^j_{(\tilde{c})} D^j(t_I)\ e^{-i\omega_I t_I} \tag{119}$$

since $D^{j=1}(t) = -\alpha(t) = 0$ for $t < -T$. Computing the integrals we obtain

$$S(\text{osc}, S_{out} \leftarrow \omega_I, \text{in}) = [\delta_{S_{out},+}\delta_{2,(a)} + \delta_{S_{out},-}\delta_{1,(a)}]\ (J^{-1})_{(a)(\tilde{c})} \left(\frac{e^{-\gamma_{(\tilde{c})}(2T)}\ e^{i\omega_I T}}{\gamma_{(b)} - i\omega_I}\ J_{(\tilde{c})(b)}\ G^{j=2}_{(b)} D^{j=2} + \frac{e^{-i\omega_I T} - e^{-\gamma_{(\tilde{c})}(2T)} e^{i\omega_I T}}{\gamma_{(\tilde{c})} - i\omega_I} \tilde{G}^j_{(\tilde{c})} D^j \right). \tag{120}$$

For large T, we can suppress the exponentially decaying terms, which reflect the fact that the oscillator modes have not completely decayed into bulk excitations. We then find that eqn. (120) can be approximated for large T as

$$[\delta_{S_{out},+}\delta_{2,(a)} + \delta_{S_{out},-}\delta_{1,(a)}]\ (J^{-1})_{(a)(\tilde{c})}\ \tilde{G}^j_{(\tilde{c})} D^j\ \frac{e^{-i\omega_I T}}{\gamma_{(\tilde{c})} - i\omega_I}. \tag{121}$$

The Bogoliubov coefficients describing production of outgoing bulk excitations generated by oscillator excitations, eqns.(115)–(116), also consist of two contributions: one from $-T$ to $+T$ during coupling and another transient after $+T$. We can rewrite as

$$\begin{aligned}
&S(\omega_F, \text{out} \leftarrow \text{osc}, S_{in}) \\
&= \int_{+T}^{+\infty} dt_F\ e^{i\omega_F t_F}\ e^{-\gamma_{(a)}(t_F - T)} E^{i=2}_{(a)} (J^{-1})_{(a)(\tilde{c})}\ e^{-\gamma_{(\tilde{c})}(2T)}\ J_{(\tilde{c})(b)}\ [\delta_{S_{in},+}\delta_{2,(b)} + \delta_{S_{in},-}\delta_{1,(b)}] \\
&+ \int_{-T}^{+T} dt_F\ e^{i\omega_F t_F}\ e^{-\gamma_{(\tilde{c})}(t_F + T)} \tilde{E}^{i=2}_{(\tilde{c})}\ J_{\tilde{c},b}\ [\delta_{S_{in},+}\delta_{2,(b)} + \delta_{S_{in},-}\delta_{1,(b)}] \\
&= \left(E^{i=2}_{(a)}\ (J^{-1})_{(a)(\tilde{c})}\ \frac{e^{-\gamma_{(\tilde{c})}(2T)}\ e^{i\omega_F T}}{\gamma_{(a)} - i\omega_F} + \tilde{E}^{i=2}_{(\tilde{c})}\ \frac{e^{-i\omega_F T} - e^{-\gamma_{(\tilde{c})}(2T)}\ e^{i\omega_F T}}{\gamma_{(\tilde{c})} - i\omega_F} \right) \\
&\qquad \times J_{(\tilde{c})(b)}\ [\delta_{S_{in},+}\delta_{2,(b)} + \delta_{S_{in},-}\delta_{1,(b)}].
\end{aligned} \tag{122}$$

For very large T the terms suppressed by an exponential factor $e^{-\gamma_{(\tilde{c})}(2T)}$ reflect a correction due to the fact that the oscillator mode has not completely been converted into bulk modes. With this correction suppressed, we find for very large T the following approximation for eqn. (122)

$$\frac{e^{-i\omega_F T}}{\gamma_{\tilde{c}} - i\omega_F} \tilde{E}^{i=2}_{(\tilde{c})}\ J_{(\tilde{c})(b)}\ [\delta_{S_{in},+}\delta_{2,(b)} + \delta_{S_{in},-}\delta_{1,(b)}] \tag{123}$$

which can be interpreted as a resonance of the oscillator for the bulk excitation produced.

In the general case (compare with the end of the previous section), the linear transformation between the “in” and the “out” operators may be expressed as

$$\begin{pmatrix} \hat{a}_{out}(\text{osc}) \\ \hat{a}^\dagger_{out}(\text{osc}) \\ \hat{a}_{out}(\omega) \\ \hat{a}^\dagger_{out}(\omega) \end{pmatrix} = \begin{pmatrix} S_{++}(\text{osc}|\text{osc}) & S_{+-}(\text{osc}|\text{osc}) & S_{++}(\text{osc}|\omega') & S_{+-}(\text{osc}|\omega') \\ S_{-+}(\text{osc}|\text{osc}) & S_{--}(\text{osc}|\text{osc}) & S_{-+}(\text{osc}|\omega') & S_{--}(\text{osc}|\omega') \\ S_{++}(\omega|\text{osc}) & S_{+-}(\omega|\text{osc}) & S_{++}(\omega|\omega') & S_{+-}(\omega|\omega') \\ S_{-+}(\omega|\text{osc}) & S_{--}(\omega|\text{osc}) & S_{-+}(\omega|\omega') & S_{--}(\omega|\omega') \end{pmatrix} \begin{pmatrix} \hat{a}_{in}(\text{osc}) \\ \hat{a}^\dagger_{in}(\text{osc}) \\ \hat{a}_{in}(\omega') \\ \hat{a}^\dagger_{in}(\omega') \end{pmatrix}$$

$$ \tag{124} $$

where summation and integration over repeated indices or variables is implied. In the case of several discrete oscillators, (osc) is replaced with an index taking integer values.

The transformation must preserve the commutation relations

$$
\begin{aligned}
[\hat{a}_{in}(\mathrm{osc}), \hat{a}^{\dagger}_{in}(\mathrm{osc})] &= 1, \\
[\hat{a}_{in}(\omega), \hat{a}^{\dagger}_{in}(\omega)] &= \delta(\omega - \omega'),
\end{aligned} \tag{125}
$$

and

$$
\begin{aligned}
[\hat{a}_{out}(\mathrm{osc}), \hat{a}^{\dagger}_{out}(\mathrm{osc})] &= 1, \\
[\hat{a}_{out}(\omega), \hat{a}^{\dagger}_{out}(\omega)] &= \delta(\omega - \omega'),
\end{aligned} \tag{126}
$$

which imply that

$$
\begin{aligned}
S_{++}(\mathrm{osc}|\mathrm{osc}) &= S^*_{--}(\mathrm{osc}|\mathrm{osc}) \\
S_{+-}(\mathrm{osc}|\mathrm{osc}) &= S^*_{-+}(\mathrm{osc}|\mathrm{osc}) \\
S_{++}(\mathrm{osc}|\omega') &= S^*_{--}(\mathrm{osc}|\omega') \\
S_{+-}(\mathrm{osc}|\omega') &= S^*_{-+}(\mathrm{osc}|\omega') \\
S_{++}(\omega|\mathrm{osc}) &= S^*_{--}(\omega|\mathrm{osc}) \\
S_{+-}(\omega|\mathrm{osc}) &= S^*_{-+}(\omega|\mathrm{osc}) \\
S_{++}(\omega|\omega') &= S^*_{--}(\omega|\omega') \\
S_{+-}(\omega|\omega') &= S^*_{-+}(\omega|\omega')
\end{aligned} \tag{127}
$$

and

$$
\begin{aligned}
&S_{++}(\mathrm{osc}|\mathrm{osc})S^*_{++}(\mathrm{osc}|\mathrm{osc}) - S_{+-}(\mathrm{osc}|\mathrm{osc})S^*_{+-}(\mathrm{osc}|\mathrm{osc}) \\
&\quad + \int_0^\infty d\omega'' \ \left[S_{++}(\mathrm{osc}|\omega'')S^*_{++}(\mathrm{osc}|\omega'') - S_{++}(\mathrm{osc}|\omega'')S^*_{++}(\mathrm{osc}|\omega'')\right] = 1,
\end{aligned} \tag{128}
$$

$$
\begin{aligned}
&S_{++}(\omega|\mathrm{osc})S^*_{++}(\omega'|\mathrm{osc}) - S_{+-}(\omega|\mathrm{osc})S^*_{+-}(\omega'|\mathrm{osc}) \\
&\quad + \int_0^\infty d\omega'' \ \left[S_{++}(\omega|\omega'')S^*_{++}(\omega'|\omega'') - S_{+-}(\omega|\omega'')S^*_{+-}(\omega'|\omega'')\right] = \delta(\omega - \omega').
\end{aligned} \tag{129}
$$

The remaining commutation relations

$$
[\hat{a}_{in}(\mathrm{osc}), \hat{a}_{in}(\mathrm{osc})] = 0, \quad [\hat{a}^{\dagger}_{in}(\mathrm{osc}), \hat{a}^{\dagger}_{in}(\mathrm{osc})] = 0, \tag{130}
$$

$$[\hat{a}_{in}(\omega), \hat{a}_{in}(\omega)] = 0, \quad [\hat{a}^\dagger_{in}(\omega), \hat{a}^\dagger_{in}(\omega)] = 0, \tag{131}$$

imply that

$$[\hat{a}_{out}(\text{osc}), \hat{a}_{out}(\text{osc})] = 0, \quad [\hat{a}^\dagger_{out}(\text{osc}), \hat{a}^\dagger_{out}(\text{osc})] = 0, \tag{132}$$

$$[\hat{a}_{out}(\omega), \hat{a}_{out}(\omega)] = 0, \quad [\hat{a}^\dagger_{out}(\omega), \hat{a}^\dagger_{out}(\omega)] = 0, \tag{133}$$

which entail the additional conditions

$$\begin{aligned} &S_{++}(\text{osc}|\text{osc})S_{+-}(\text{osc}|\text{osc}) - S_{+-}(\text{osc}|\text{osc})S_{++}(\text{osc}|\text{osc}) \\ &\quad + \int_0^\infty d\omega'' \ [S_{++}(\text{osc}|\omega'')S_{+-}(\text{osc}|\omega'') - S_{+-}(\text{osc}|\omega'')S_{++}(\text{osc}|\omega'')] = 0, \end{aligned} \tag{134}$$

$$\begin{aligned} &S_{++}(\omega|\text{osc})S_{+-}(\omega'|\text{osc}) - S_{+-}(\omega|\text{osc})S_{++}(\omega'|\text{osc}) \\ &\quad + \int_0^\infty d\omega'' \ [S_{++}(\omega|\omega'')S_{+-}(\omega'|\omega'') - S_{+-}(\omega|\omega'')S_{++}(\omega'|\omega'')] = \delta(\omega - \omega'). \end{aligned} \tag{135}$$

Similarly, we can construct the number operators for quanta in the "out" space, namely $N_{out}(\text{osc}) = \hat{a}^\dagger_{out}(\text{osc})\hat{a}_{out}(\text{osc})$ for the number of oscillator quanta and $N_{out}(\omega) = \hat{a}^\dagger_{out}(\omega)\hat{a}_{out}(\omega)$ for the number of bulk quanta of frequency ω. We find the corresponding expectation values

$$\begin{aligned} \langle 0_{in}|\hat{N}_{out}(\text{osc})|0_{in}\rangle &= |S_{+-}(\text{osc}|\text{osc})|^2 + \int_0^\infty d\omega' |S_{+-}(\text{osc}|\omega')|^2 && (136) \\ \langle 0_{in}|\hat{N}_{out}(\omega)|0_{in}\rangle &= |S_{+-}(\omega|\text{osc})|^2 + \int_0^\infty d\omega' |S_{+-}(\omega|\omega')|^2, && (137) \end{aligned}$$

which will vanish if the "in" and "out" vacua coincide.

In the application to braneworld cosmology, when we observe the cosmological perturbations today, we will (using the assumption of Gaussianity) be measuring, in effect, expectation values of observables quadratic in the creation and annihilation operators localized on the brane today, in other words, the "out" oscillator operators. Ignoring the additional discrete indices (which we suppress), we offer a useful parameterization of the relevant part of the S matrix. Let $a_{brane,out}$ and $a^\dagger_{brane,out}$ be the operators of which we want to take the quadratic expectation value. The S-matrix expresses these as a linear combination of $a_{brane,out}$ and $a^\dagger_{brane,out}$, on the one hand, and of $a_{bulk,in}(\omega)$ and $a^\dagger_{bulk,in}(\omega)$, on the other. The following offers a useful parameterization of this transformation. Require

that $A_{brane,in}$ and $A_{bulk,in}$, normalized such that $[A_{brane,in}, A^\dagger_{brane,in}] = [A_{bulk,in}, A^\dagger_{bulk,in}] = 1$, be entirely on the brane and in the bulk, respectively. Then $a_{brane,out}$ may be expressed in terms of these according to one of the three following possibilities: either

$$a_{brane,out} = \cos\theta\ A_{brane,in} + \sin\theta\ A_{bulk,in} \tag{138}$$

where $0 \le \theta \le \pi/2$; or ,

$$a_{brane,out} = \cosh\xi\ A_{brane,in} + \sinh\xi\ A^\dagger_{bulk,in} \tag{139}$$

where $0 \le \xi \le +\infty$; or

$$a_{brane,out} = \sinh\xi\ A^\dagger_{brane,in} + \cosh\xi\ A_{bulk,in} \tag{140}$$

where $0 \le \xi \le +\infty$. We observe that the bulk initial state may have a very important, even dominant, role in determining what we see on the brane today. $A_{brane,in}$, and its conjugate, may in turn be related to $a_{brane,in}$ by a Bogoliubov transformation far from the identity, and the same applies to $A_{bulk,in}$.

IV. GENERALIZATION TO NONTRIVIAL DYNAMICS ON THE STRING: REFLECTION, DIFFRACTION, AND DISPERSION

In the previous section, because the undulations on the string were described by a massless wave equation, it was possible to separate $u(t,x)$ on the string into a sum of the form $u_{in}(t+x) + u_{out}(t-x)$. The left-moving component $u_{in}(t)$ acted as a source for the oscillator, which in turn radiated exclusively into the right-moving u_{out} channel. However, for a massive wave equation of the form

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + m^2(x)\right] u(t,x) = 0, \tag{141}$$

it is no longer possible to separate into left-moving and right-moving components. If we consider the $m^2(x)u(t,x)$ term as a perturbation, we see that initially right-moving outgoing waves are scattered back onto the oscillator situated on the boundary, and incident left-moving waves, rather than striking the oscillator, are sometimes rescattered back toward infinity by this term. In the braneworld problem, the curved geometry, due to the

fact that the bulk spacetime is AdS or AdS-like rather than Minkowski, introduces such scattering. For the time-independent problem, all these effects can be treated straightforwardly by decomposing into Fourier modes of definite frequency. Nothing more is required than solving eqn. (141) by separation of variables with the appropriate boundary conditions, following the formalism developed in section II. However, for the time-dependent problem, when no such decomposition is possible, it is necessary to solve for these multiple reflections in real space rather than in momentum space. One possibility would be to treat the $m^2(x)u(t,x)$ term as a perturbation that generates an infinite series expansion in the number of reflections. However, for all but some very special cases, this series is either divergent or very slowly convergent.

A well-behaved series may be obtained by noting that for all frequencies except those near the resonance $\omega \approx \omega_0$, there is very little interaction between the string and the oscillator. Setting

$$u_{out}(\omega) = P(\omega)\ u_{in}(\omega), \tag{142}$$

we observe that

$$P(\omega) = (-)\cdot\frac{\omega^2 - \bar{\omega}^2 - i\gamma\omega}{\omega^2 - \bar{\omega}^2 + i\gamma\omega}. \tag{143}$$

Away from the resonance, the end of the string acts almost exactly as a Dirichlet boundary condition, for which one would have $P(\omega) = -1$ exactly. Consequently, to obtain a rapidly convergent series, it is advantageous as a first approximation to solve for the propagation as if there were Dirichlet boundary conditions. Then the violation of the boundary condition for the oscillator becomes a source from which a perturbative expansion by successive approximations can be generated.

More explicitly, for a given incoming wave, with Cauchy initial data specified on past null infinity $\mathcal{N}^{(-)}$, we first find a continuation $u_0(t,x)$ such that the Dirichlet boundary condition

$$u_0(t, x = 0) = 0 \tag{144}$$

is satisfied. We shall call the $u_0(t,x)$ obtained in this way the "incident wave." We then construct a series expansion

$$u(t,x) = \sum_{n=0}^{+\infty} u_n(t,x) \tag{145}$$

such that

$$\left[\frac{d^2}{dt^2}+\bar{\omega}^2(t)\right]u(t,x=0)=\gamma\left.\frac{\partial u}{\partial x}\right|_{x=0} \tag{146}$$

is satisfied. Here the string-oscillator coupling γ (equal to τ/m where τ is the string tension) serves as the expansion parameter. We correct for the violation of eqn. (146) by $u_0(t,x=0)$ by choosing boundary data for u_1 such that

$$\left[\frac{d^2}{dt^2}+\bar{\omega}^2(t)\right]u_1(t,x=0)=\gamma\left.\frac{\partial u_0}{\partial x}\right|_{x=0} \tag{147}$$

by means of a retarded Green's function for the oscillator. Then this Dirichlet boundary data for u_1 on the boundary is used to extend u_1 to the full domain $x\geq 0$ using the Dirichlet form of the retarded Green's function in the bulk, in other words

$$u_1(t,x)=\int_{-\infty}^{t}dt'\frac{\partial}{\partial x'}G_D(t,x;t',x'=0)\ u_1(t',x'=0). \tag{148}$$

This process repeats itself, so that the relations

$$\left[\frac{d^2}{dt^2}+\bar{\omega}^2(t)\right]u_{n+1}(t,x=0)=\gamma\left.\frac{\partial u_n}{\partial x}\right|_{x=0} \tag{149}$$

and

$$u_{n+1}(t,x)=\int_{-\infty}^{t}dt'\frac{\partial}{\partial x'}G_D(t,x;t',x'=0)\ u_{n+1}(t',x'=0) \tag{150}$$

are satisfied. At the boundary we use the Green's function for the oscillator to obtain

$$u_{n+1}(t,x=0)=\int_{-\infty}^{t}dt'\ G_{osc}(t,t')\ \gamma\frac{\partial u_n}{\partial x'}(t',x'=0). \tag{151}$$

In terms of propagators, we may write

$$\begin{aligned}
u(t,x) &= u_0(t,x)\\
&+\ \gamma\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt^{(2)}\ D_{x'}G_{bulk}(t,x;t',x'=0)\ G_{osc}(t';t^{(2)})\ D_{x^{(2)}}u_0(t^{(2)},x^{(2)}=0)\\
&+\ \gamma^2\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt^{(2)}\ \int_{-\infty}^{t^{(2)}}dt^{(3)}\ \int_{-\infty}^{t^{(3)}}dt^{(4)}\\
&\quad\times D_{x'}G_{bulk}(t,x;t',x'=0)\ G_{osc}(t';t^{(2)})\ D_{x^{(2)}}D_{x^{(3)}}G_{bulk}(t^{(2)},x^{(2)}=0;t^{(3)},x^{(3)}=0)\\
&\quad\times G_{osc}(t^{(3)};t^{(4)})D_{x^{(4)}}u_0(t^{(4)},x^{(4)}=0)\\
&+\ \ldots
\end{aligned} \tag{152}$$

This expansion contains the infinite sum

$$
\begin{aligned}
\bar{G}_{osc} &= G_{osc} \\
&+ \gamma G_{osc}(D_{xI} D_{xII} G_{bulk}) G_{osc} \\
&+ \gamma^2 G_{osc}(D_{xI} D_{xII} G_{bulk}) G_{osc}(D_{xI} D_{xII} G_{bulk}) G_{osc} \\
&+ \gamma^3 G_{osc}(D_{xI} D_{xII} G_{bulk}) G_{osc}(D_{xI} D_{xII} G_{bulk}) G_{osc}(D_{xI} D_{xII} G_{bulk}) G_{osc} + \ldots
\end{aligned} \tag{153}
$$

Here the notation D_{xI} and D_{xII} indicates that the spatial derivative acts on the spatial component of the first and the second argument of G_{bulk}, respectively. We may decompose the internal bulk propagator lines into a singular and a regular parts

$$
D_{xI} D_{xII} G_{bulk} = K_{bulk}^{(reg)} + K_{bulk}^{(sing)}. \tag{154}
$$

Consider for example the massless propagator

$$
[\partial_t^2 - \partial_x^2]^{-1}(t, x; t', x') = (1/2)\ \theta(t - t')\ \theta\Big((t - t')^2 - (x - x')^2\Big), \tag{155}
$$

or the uniform mass propagator

$$
\begin{aligned}
&[\partial_t^2 - \partial_x^2 + m^2]^{-1}(t, x; t', x') = \\
&\quad (1/2)\ \theta(t - t')\ \theta\Big(\sqrt{(t - t')^2 - (x - x')^2}\Big) J_0\Big(m\sqrt{(t - t')^2 - (x - x')^2}\Big).
\end{aligned} \tag{156}
$$

These are the propagators for the entire infinite plane $\mathcal{R}^2$. We form the retarded Dirichlet propagator by using the method of images so that

$$
G_D(t, x; t', x') = G_\infty(t, x; t', x') - G_\infty(t, x; t', -x'). \tag{157}
$$

It turns out that the singular (local) part has the form

$$
K_{bulk}^{(sing)}(t - t') = -\delta'(t - t'), \tag{158}
$$

no matter what the propagator is, that is, irrespective of the particular form of the function $m^2(x)$ in eqn. (141). By contrast, $K_{bulk}^{(reg)}$ depends sensitively on the particular form of $m^2(x)$ and encodes all the non-locality arising from propagation from the brane into the bulk and back again.

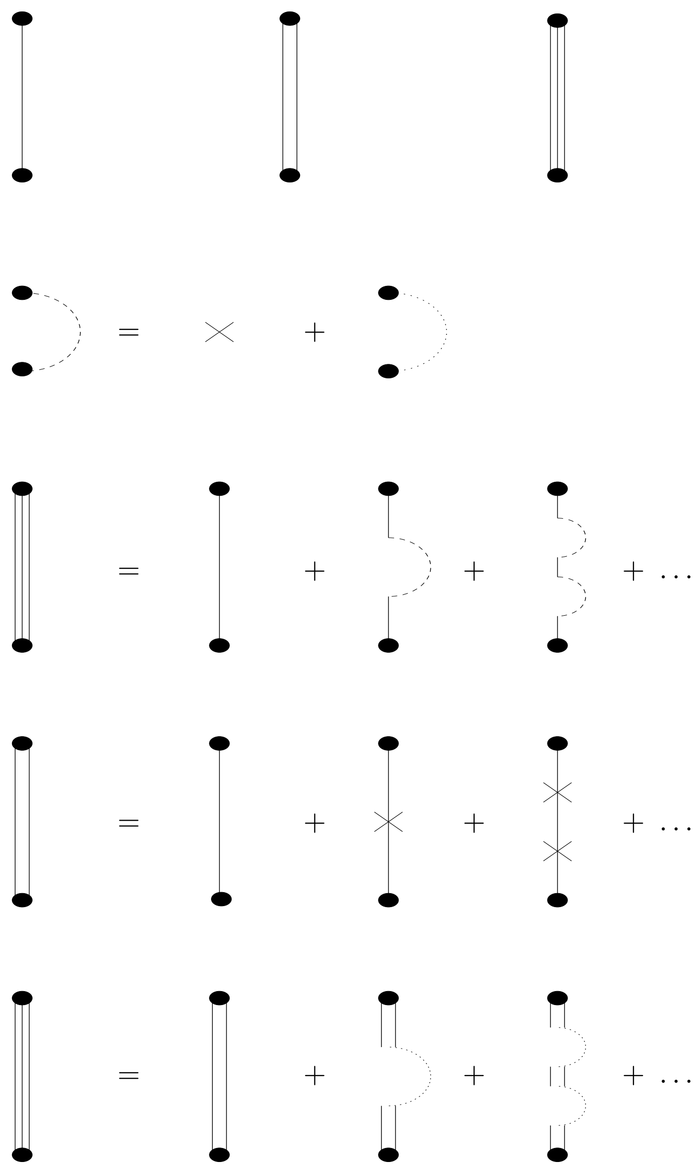

FIG. 4: **Perturbation expansion for the full brane-brane two-point function as a power series in the brane-bulk coupling** γ. The first line shows three increasingly complete forms for the brane-brane propagator. The most complete is the full propagator, denoted by the triple line, including both dissipative and nonlocal effects due to the interaction with the bulk. The double line includes only the dissipative effects of the bulk interaction, and the single line does not include any correction at all from the brane-bulk interactions. For the simplest case, the single line represents $(D_t^2 + \omega_0^2)^{-1}$ and the double line $(D_t^2 + \gamma D_t + \omega_0^2)^{-1}$. More generally, ω_0 and γ may depend on time. On the second line we show the diagrammatic decomposition of the full bulk-bulk propagator K (with both endpoints on the brane and the brane-bulk coupling factors included), represented by the dashed curve. This propagator decomposes into its purely local singular part, represented by the cross, and its nonlocal, regular part, represented by the dotted line. The third line illustrates the perturbative expansion for the full brane propagator. On the fourth line the partial sum over the local insertions is carried out. The last line shows the expansion of the full propagator in terms of the dissipation corrected propagator and the nonlocal corrections.

We use the massless propagator

$$G_D(\underline{\mathbf{x}}, \underline{\mathbf{x}}') = G_\infty(\underline{\mathbf{x}}, \underline{\mathbf{x}}') - G_\infty(\underline{\mathbf{x}}, \underline{\mathbf{x}}'^R) \tag{159}$$

because the correction due to mass or, more generally, any potential contributes only to the nonlocal part of K_{bulk} where t and t' do not coincide.

We have

$$\begin{aligned} K(t,t') &= \partial_x \partial_{x'} G_D(t,x;t',x')\Big|_{x=x'=0} \\ &= \frac{1}{2} \lim_{x\to 0+} \left[\lim_{x'\to 0+} \partial_x \partial_{x'} \Big(\theta(t-t') \cdot \Big\{ \theta(t-t'+x-x') - \theta(t-t'-x-x') \Big\} \Big) \right] \\ &= \frac{1}{2} \lim_{x\to 0+} \left[\lim_{z'\to 0+} \Big\{ -\delta'(t-t'+x-x') - \delta'(t-t'-x-x') \Big\} \right] \\ &= -\delta'(t-t'). \end{aligned} \tag{160}$$

The order of the limits is arbitrary; however, by choosing a certain order as we have done here, we reduce the number of terms to consider.

We now consider the contribution for $t > t'$ where we have used the massive propagator

$$G_\infty(\underline{\mathbf{x}}, \underline{\mathbf{x}}') = \frac{1}{2}\theta(t-t')\ \theta(\tau) J_0(m\tau), \tag{161}$$

where

$$\tau = \sqrt{(t-t')^2 - (z-z')^2}. \tag{162}$$

One has

$$\begin{aligned} K_{reg}(t-t') &= \frac{1}{2} \frac{\partial}{\partial x} \frac{\partial}{\partial x'} \left[J_0\Big(m\sqrt{(t-t')^2 - (x-x')^2}\Big) - J_0\Big(m\sqrt{(t-t')^2 - (x+x')^2}\Big) \right] \\ &= mJ_0'\Big(m(t-t')\Big) \frac{\partial}{\partial x} \frac{\partial}{\partial x'} \sqrt{(t-t')^2 - (x-x')^2}\Big|_{x=x'=0} \\ &= -\frac{mJ_0'\Big(m(t-t')\Big)}{(t-t')}. \end{aligned} \tag{163}$$

We observe that the regular part here is in fact regular as $t \to t'+$, since $J_0'(z) \sim z$. This is a universal feature.

Since

$$G_{osc} = \frac{1}{D_t^2 + \bar{\omega}^2(t)}, \tag{164}$$

it follows that

$$\begin{aligned}\hat{G}_{osc} &= G_{osc} + G_{osc}\Big(\gamma K^{(sing)}_{bulk}\Big)G_{osc} + G_{osc}\Big(\gamma K^{(sing)}_{bulk}\Big)G_{osc}\Big(\gamma K^{(sing)}_{bulk}\Big)G_{osc} + \dots \\ &= \frac{1}{D_t^2 + \gamma D_t + \bar{\omega}^2(t)}\end{aligned} \tag{165}$$

and

$$\bar{G}_{osc} = \hat{G}_{osc} + \hat{G}_{osc}\Big(\gamma K^{(reg)}_{bulk}\Big)\hat{G}_{osc} + \hat{G}_{osc}\Big(\gamma K^{(reg)}_{bulk}\Big)\hat{G}_{osc}\Big(\gamma K^{(reg)}_{bulk}\Big)\hat{G}_{osc} + \dots. \tag{166}$$

The above expansions may be interpreted pictorially by means of Feynman diagrams, as indicated in Fig. 4. The propagator for the oscillator with the coupling to the bulk ignored given in eqn. (164) is denoted by a single vertical line localized on the brane. The dashed curves represent propagation of bulk gravitons and their coupling to the brane. We decompose the bulk propagator into two parts: a purely local singular part, denoted by a cross insertion, and a regular, nonlocal part denoted by a dotted curve, as indicated on the second line of the figure. The third line shows how the full two-point function on the brane (indicated by the triple line) may be expressed as an infinite sum. The fourth line indicates a partial infinite sum with only the local insertions included yielding an improved propagator on the brane (denoted by by the double vertical line). In terms of the effective equation on the brane, this infinite summation simply represents the addition of a local dissipation term to the oscillator equation of motion. Finally, on the last line, an infinite sum adding the nonlocality is taken to calculate the exact full two-point function on the brane.

V. GENERALIZATION TO MOVING BOUNDARIES

For computing braneworld cosmological perturbations, we are interested in systems with a moving boundary. In the picture where Birkhoff's theorem is exploited so that the bulk is stationary and the brane moves, except for the boundary condition imposed by the moving brane, the propagation in the bulk is trivial. One simply propagates the bulk gravitons using the coordinate system that best exploits the symmetries of AdS.

For the scalar wave equations considered here, the closest equation to the case of AdS gravitons is

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + k^2 + \frac{15/4}{x^2}\right] u(t,x) = 0, \tag{167}$$

where k corresponds to the magnitude of the wave number in the three transverse spatial dimensions. The boundary is described by a trajectory $x = x_b(t_b)$, which may be parameterized by proper time τ where

$$\frac{d\tau}{dt_b} = a(x_b) \cdot \sqrt{1 - {x'_b}^2} \tag{168}$$

and $a(x_b)$ is the conformal scalar factor of the metric. We also define a barred proper time $\bar{\tau}$ where this conformal factor has been omitted. The oscillator is treated as in the previous section except that now the proper time τ takes the place of ordinary time. We parameterize $\underline{\mathbf{x}}(\tau) = \Big(t_b(\tau), x_b(\tau)\Big)$. Most everything discussed in the previous section for the case of a stationary boundary generalizes straightforwardly to the case of a moving boundary. The only slight complication is the computation of the Dirichlet Green's functions and the behavior of the u_{in} and u_{out} solutions, which are represented by the external lines in our diagrammatic expansion. However, this may be accomplished by placing a virtual source an infinitesimal distance just beyond the boundary. This method, which is a sort of generalization of the method of images, is originally due to d'Alembert.[11]

For the external lines in our diagrammatic expansion in the oscillator/bulk coupling strength γ, we seek two complete bases of "in" and "out" states, characterized by the asymptotic behavior on past and future null infinity, respectively. The solutions satisfy

$$\mathcal{L}_{bulk} u(t,x) = \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + k^2\right] u(t,x) = 0 \tag{169}$$

where u can be either u_{in} or u_{out} and satisfy the *hard* boundary condition

$$\mathcal{L}_{B,hard} u = 0 \tag{170}$$

on the boundary $\underline{\mathbf{x}} = \underline{\mathbf{x}}_b(\bar{\tau})$.

By *hard* we mean the condition that is local on the boundary that agrees with the exact boundary condition (which we shall call "soft") in the short-distance or high frequency limit. For the examples considered in this paper this is a Dirichlet boundary condition, where $u = 0$ on the boundary, but Neumann and mixed boundary conditions are allowed, because these do not relate the behavior at different points on the boundary in a nonlocal way.

For a stationary boundary the asymptotic states u_{out} and u_{in} may be readily constructed by separation of variables. For a moving boundary accounting for multiple reflections is less trivial.

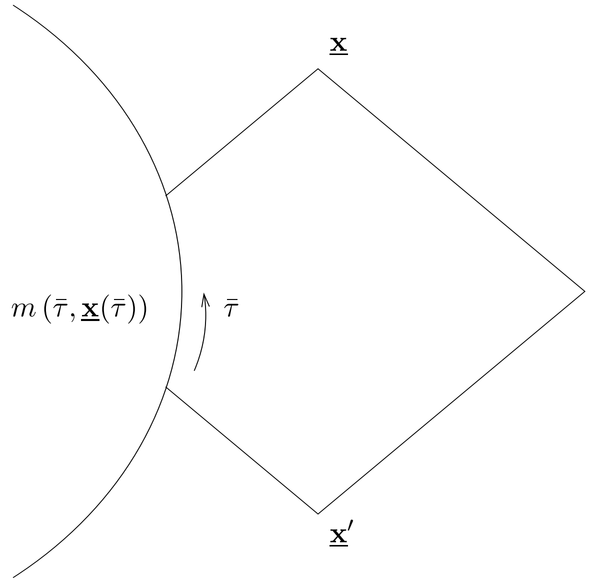

FIG. 5: **Method of virtual sources.** We illustrate how the bulk propagator from $\underline{\mathbf{x}}'$ to $\underline{\mathbf{x}}$ may be calculated for a moving boundary using the method of virtual sources. In general we do not know the form of the propagator satisfying the relevant boundary conditions. Therefore, to compute $G(\underline{\mathbf{x}}, \underline{\mathbf{x}}')$, we start with an arbitrary propagator, in general violating the boundary conditions, and then place a virtual source just infinitesimally beyond the boundary to correct for this violation. Here a corrective virtual source is required on that portion of the boundary intersecting with both the forward lightcone of $\underline{\mathbf{x}}'$ and the past lightcone of $\underline{\mathbf{x}}$, which we parameterize by $\bar{\tau}$. This virtual source (for the case of Dirichlet boundary conditions) is a dipole layer, whose strength as a function of position on the boundary is determined by solving a Volterra integral equation of the second kind.

Before calculating the extensions of u_{in} and u_{out} based on their behaviors in the neighborhood and past and future null infinity, respectively, we first construct the retarded Green's function satisfying the Dirichlet boundary condition on the null boundary

$$G_D(\underline{\mathbf{x}}, \underline{\mathbf{x}}') \tag{171}$$

starting from the retarded Green's function that would hold if there were no boundary

$$G_\infty(\underline{\mathbf{x}}, \underline{\mathbf{x}}'). \tag{172}$$

We proceed by postulating a virtual source just beyond the boundary, propagated according to G_∞, which serves to enforce the boundary conditions on the moving boundary. For Dirichlet boundary conditions this is a dipole layer of strength characterized by the linear density $m(\bar{\tau})$ where the dipole is inward and normally directed. The field on the boundary of a pointlike dipole of unit strength at τ' just beyond the boundary is of the

form

$$\frac{1}{2}\delta(\bar{\tau}-\bar{\tau}')+\theta(\bar{\tau}-\bar{\tau}')R(\bar{\tau},\bar{\tau}') \tag{173}$$

where $R(\tau,\tau')$ is a smooth, regular function without any divergence at $\bar{\tau}=\bar{\tau}'$. [For the case of Neumann or mixed boundary conditions, we would use a monopole rather than a dipole layer.]

We write as an ansatz

$$G_D(\underline{\mathbf{x}},\underline{\mathbf{x}}')=G_\infty(\underline{\mathbf{x}},\underline{\mathbf{x}}')+\int_{\bar{\tau}_a}^{\bar{\tau}_b} d\bar{\tau}\ (\hat{n}(\bar{\tau})\cdot\nabla_2)\ G_\infty\Big(\underline{\mathbf{x}},\underline{\mathbf{x}}(\bar{\tau})\Big)\ m(\bar{\tau};\underline{\mathbf{x}},\underline{\mathbf{x}}') \tag{174}$$

as indicated in Fig. 5. Here $\hat{n}(\bar{\tau})$ is the inward normal vector on the boundary and ∇_2 denotes derivative with respect to the second argument of the Green's function. Here the interval $[\tau_a,\tau_b]$, which can be empty for certain choices of $\underline{\mathbf{x}}$, $\underline{\mathbf{x}}'$, is the intersection of the boundary worldline, the interior of the forward lightcone of $\underline{\mathbf{x}}'$, and the interior of the past lightcone of $\underline{\mathbf{x}}$. The virtual source $m(\bar{\tau};\underline{\mathbf{x}},\underline{\mathbf{x}}')$ is chosen so that the boundary conditions are satisfied. This is accomplished by solving the integral equation

$$\frac{-1}{2}m\Big(\bar{\tau};\underline{\mathbf{x}},\underline{\mathbf{x}}'\Big)=G_\infty\Big(\underline{\mathbf{x}}(\bar{\tau}),\underline{\mathbf{x}}'\Big)+\int_{\bar{\tau}_a}^{\bar{\tau}} d\bar{\tau}'\ (\hat{n}(\bar{\tau})\cdot\nabla_2)\ G_\infty\Big(\underline{\mathbf{x}}(\bar{\tau}),\underline{\mathbf{x}}(\bar{\tau}')\Big)\ m(\bar{\tau}';\underline{\mathbf{x}},\underline{\mathbf{x}}'). \tag{175}$$

It is understood that the singular part, which occurs on the left-hand side of the equation, has been removed from the integral kernel. This is a Volterra integral equation of the second kind, which is always guaranteed to have a unique solution [12]. The "in" and "out" functions may be constructed analogously using an integral equation.

VI. DISCUSSION

One of the most interesting questions of braneworld cosmology is the nature of the cosmological perturbations predicted and how these might differ from those possible in models not having a large extra dimension. In order to carry out the required calculation, the brane-bulk interaction must be taken into account, preferably without resorting to approximations based on lower-dimensional effective descriptions. The techniques developed in this paper, illustrated using scalar toy models in which a (1+1)-dimensional scalar field theory is coupled to a finite number of degrees of freedom on its boundary, constitute a

step toward achieving such a complete calculation and demonstrate some of the qualitative effects that one might expect.

In section II we studied some simple time-independent systems. It was seen how coupling the boundary (brane) degrees of freedom to the continuum (bulk) gave rise to *dissipative* effects from a purely brane point of view. In addition, lack of spatial homogeneity in the continuum (as would result from the "warp factor" of Randall-Sundrum-like cosmologies) led to *nonlocal* effects, which can be encoded into a spectral density $\rho(\omega)$. In section III we saw how new effects could arise in time-dependent systems (as would arise in an expanding, rather than static Minkowski, geometry on the brane). The discrete degrees of freedom on the brane, rather than being coupled at the same strength infinitely far into the past, could initially be uncoupled, interacting effectively for only a finite amount of time, so that a sort of S matrix linking the "in" oscillator states to the "out" oscillator states for the brane and bulk dynamics would connect observables today with initial conditions for the brane and the bulk. When we observe the braneworld perturbations today, for the simplest case where these are Gaussian, it suffices to calculate the expectation values of operators quadratic in $a_{brane,out}$ and $a^{\dagger}_{brane,out}$. [Note that our simplified notation here assumes that the wave number in the three transverse spatial dimensions $\mathbf{k}$ has been fixed and ignores the fact that these operators also have an integer index to account for the possibility of several modes on the brane. Similarly, the bulk operators have a discrete index to account for the several bulk graviton polarizations and other possible modes propagating in the bulk. Since the generalization is straightforward, we suppress these complication in the notation that follows.] In terms of the "in" state, $a_{brane,out}$ and $a^{\dagger}_{brane,out}$ may be expressed as a linear combination of "in" operators, $A_{brane,in}$ and $A_{bulk,in}$, normalized so that

$$[A_{brane,in}, A^{\dagger}_{brane,in}] = 1, \tag{176}$$

and

$$[A_{bulk,in}, A^{\dagger}_{bulk,in}] = 1, \tag{177}$$

where $A_{brane,in}$ is constructed entirely as a linear combination of $a_{brane,in}$ and $a^{\dagger}_{brane,in}$, and likewise $A_{bulk,in}$ is constructed entirely as a linear combination of $a_{bulk,in}(\omega)$ and $a^{\dagger}_{bulk,in}(\omega)$, where $\omega \geq 0$. There are three possibilities: firstly,

$$a_{brane,out} = \cos\theta \ A_{brane,in} + \sin\theta \ A_{bulk,in} \tag{178}$$

where $0 \le \theta \le \pi/2$; secondly,

$$a_{brane,out} = \cosh\xi\ A_{brane,in} + \sinh\xi\ A^{\dagger}_{bulk,in} \tag{179}$$

where $0 \le \xi \le +\infty$; and thirdly,

$$a_{brane,out} = \sinh\xi\ A^{\dagger}_{brane,in} + \cosh\xi\ A_{bulk,in} \tag{180}$$

where $0 \le \xi \le +\infty$. Physically, when we measure the perturbations today, we observe in part the character of the initial state on the brane and in part that of the initial state of the bulk. Their relative importance can be read off from this S-matrix.

In section IV we dealt with reflections in the bulk for the time-dependent situation. Here a wave initially propagating away from the brane is scattered, reflected, or diffracted—depending on which term one prefers—by the bulk so as to propagate back toward the brane, and vice versa. A perturbation expansion was developed where "hard" boundary conditions are imposed at zeroth order and the "soft" corrections are included by successive approximations. A "hard" boundary condition is defined as one that is completely local on the boundary, such as the Neumann or Dirichlet boundary condition, or a mixture of the two. These are "non-dynamical" because they do not interrelate the boundary behavior at distinct points on the boundary. By contrast, "soft" boundary conditions are dynamical and nonlocal, resulting when the degrees of freedom on the boundary cede to the incoming waves. Asymptotically, away from the resonances of the degrees of freedom on the boundary, the exact "soft" boundary conditions approach a corresponding "hard" boundary condition, which we use at zeroth order in our perturbation expansion.

In section V we indicated how to deal with the complication of a moving boundary. In most respects, the techniques of the previous sections generalize straightforwardly. The time for the degrees of freedom on the boundary becomes proper time. The principal difficulty in carrying out the generalization to a moving boundary having an arbitrary trajectory (corresponding to an arbitrary expansion history for braneworld cosmology) is the calculation of the bulk Green's functions satisfying the hard boundary conditions. This is accomplished by the method of virtual sources, where an arbitrary Green's function (in general not respecting the boundary condition) is used and then a virtual source placed just infinitesimally behind the boundary is postulated in order to emit into the bulk waves

that correct for the violation of the boundary condition due to using the wrong Green's function. The source that achieves such a correction is calculated by solving an integral equation of the Volterra form of the second kind.

We are in the process of generalizing this work to include gravitational gauge fixing so that it can be applied to calculating braneworld cosmological perturbations to linear order. This work will be the subject of a future publication.[13]

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